DOCUMENT RESUME

ED 213 576 SE 035 603

AUTHOR Watson, F. R.
TITLE Exploring Numbers: Some Investigations with a Calculator. [Series A, B, and C, Teachers' Notes and Pupils' Activities].
INSTITUTION Keele Univ. (England). Inst. of Education.
PUB DATE Feb 79
NOTE 93p.; Not available in hard copy due to marginal legibility of original document.
AVAILABLE FROM Information Reference Center (ERIC/IRC), The Ohio State Univ., 1200 Chambers Rd., 3rd Floor, Columbus, OH 43212 ($3.50)

EDRS PRICE MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS *Calculators; Discovery Learning; Educational Technology; Instructional Materials; *Learning Activities; Mathematics Instruction; Mathematics Materials; *Number Concepts; Problem Solving; Secondary Education; *Secondary School Mathematics; Supplementary Reading Materials; Teaching Guides; Worksheets

ABSTRACT

These documents are three separate sets of student problems and teaching guides geared towards calculators. Series A is designed for pupils from ages 10 to 13; "B" is structured for 13 through 16, and "C" for 16 through 19. The materials are designed to provide activities for investigation and exploration of number properties and patterns that are not found in most current textbooks. It is hoped that these materials will be a useful starting point for teachers, who should feel free to modify and expand them for their own needs. (MP)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED213576

"PERMISSION TO REPRODUCE THIS MATERIAL IN MICROFICHE ONLY HAS BEEN GRANTED BY
F.R. WATSON
TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

EXPLORING NUMBERS

Some Investigations with a Calculator

Series A

TEACHERS' NOTES

U.S. DEPARTMENT OF EDUCATION
NATIONAL INSTITUTE OF EDUCATION
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)
✓ This document has been reproduced as received from the person or organization originating it.
Minor changes have been made to improve reproduction quality.
• Points of view or opinions stated in this document do not necessarily represent official NIE position or policy.

F.R. Watson
Institute of Education
University of Keele

SE 035 603

Copyright © F.R. Watson, University of Keele, 1979

Note. The reproduction of Pupil material by teachers for use with their own pupils is expressly permitted.

The advent of the 'pocket' calculator seems likely to have a profound effect on mathematics teaching before long. One way in which it can be used immediately, without any change in 'syllabus', is as a means of exploration and investigation of number properties and patterns.

It is hoped these suggestions will be a useful starting point for teachers, who will modify and expand them to suit their own circumstances. I should welcome any comments, and notes of errors which, perhaps inevitably, will have escaped my attention.

Several people have helped in the production of these materials, but I should particularly like to thank Mrs. B. Wiggins and Mrs. C. Lea for their contributions in typing and printing respectively.

F.R. Watson
February 1979

Exploring Numbers: Number patterns and investigations:
Teacher's notes and solutions for Series A (10-13 years)

Some teachers of this age group may have misgivings about allowing children to use calculators at all in mathematics, other than for checking pencil-and-paper work, fearing that they may thereby lose all arithmetical facility. These fears, though real, may be unjustified; if children are to use the calculator effectively, they will still need to be proficient in single digit arithmetic and in estimation ("4.3 x 7.2 cannot be 0.59722 since the answer is about 4 x 7 = 28") and it may well be that if left to their own choice children will naturally reach an "equilibrium" position in which simple calculations are done mentally, and the more complex ones done by calculator.

However, this controversy is incidental to the purpose of these notes, which is to suggest the use of the calculator as a 'laboratory' for the exploration and investigation of number properties. Certainly it is not the intention of these exercises to weaken children's arithmetical skills, but rather to provide purposeful and interesting situations for calculation, which also illustrate another facet of mathematics - that of the study of patterns.

Sometimes the results of the investigations are themselves important and provide a foundation for work which will be undertaken later on in mathematics. In other cases, this is not so - but though the 'final answer' may not be an important mathematical theorem, the exploration itself may have useful side-effects, such as providing practice, awakening interest, etc. - quite apart from its value in demonstrating the nature of mathematics as "a process of finding out". The importance of this aspect of mathematics teaching, which I believe to be unduly neglected currently - has been argued elsewhere, and the interested reader may refer to "M = C + P: Some thoughts on attainment in Mathematics", Mathematics in Schools, 7, 4, September 1978.

At this stage children have some facility in dealing with the arithmetic of whole numbers, but the use of the calculator may increase confidence and motivation, particularly if knowledge of tables and 'addition bonds', and general number manipulation, is rather shaky. The work may also be suitable for older children who have found arithmetic difficult or impossible, and some of these ideas have been tried out, using calculators, with poorly-motivated fourth year secondary pupils whose arithmetical skills were very weak.

The calculator may ease the burden of computation, illustrate interesting patterns in number work, and perhaps rekindle some interest in mathematics. It is hoped that with increasing confidence and practice, children's performance in arithmetic without the calculator will also improve, and this aim should be borne in mind throughout - e.g. by encouraging pupils to attempt the simpler examples by mental arithmetic as their confidence grows.

Activities for Pupils - a list of the materials.

1. Patterns using differences.
2. $(n+1)(n-1)$, $(n+5)(n-5)$.
3. Multiples of 9.
4. Powers of 11, and of 101, leading to Pascal's Δ.

5, 6, 7. Δ and square numbers.

8. An arrow game. $u_{n+1} = \begin{cases} \frac{1}{2}u_n \\ 3u_n+1 \end{cases}$

(+) 9. Patterns of 'units' digits.

10. Factors.
11. Divisibility by 2, 3, 9.
12. Prime numbers.
13. Numbers which can be written as x^2+y^2.
14. Common factors. H. C. F.
15. Multiples. L. C. M.

(+) 16. Limited use of keys (⊕ and ⊗).

17. Pascal's Δ.

18.19.20. Puzzles.

(+) 21. Inverse operations ⊕ , ⊖

22. a+(b+c), a+b=b+a.

23. a×(b×c), a×b=b×a.

24. a(b+c).

25. (b+c)a.

26. Conventions - order of applying +, -, ×, ÷.

(Those marked + are of double length)

Notes on the Number pattern investigations - series A.

It is suggested that these should be consulted after you have tried the corresponding problems. This gives a better idea of likely pupil difficulties and may also make these brief notes more comprehensible. Some of the ideas in each question are mentioned briefly and answers or notes are also given. Though the calculator is not essential in some questions, it will be useful for testing the hypotheses using large numbers - and pupils can also CHECK their working using the calculator.

1 (a, b, c) Ideas of tables, even/odd, sequence (pattern), differences between successive terms, noticing and predicting regularities.

1 (d) 3n, 3n+1, 3n+2 (c.f. modulo arithmetic)
1, 3, 6, 10 are Δ numbers (see Q. 5) with successive differences 1, 2, 3, 4...
1, 4, 9, 16 are square numbers with successive differences the odd numbers (and see Q. 6).

2 (a) "n^2 is 1 more than (n-1)(n+1)", i.e. $n^2-1 = (n-1)(n+1)$

(b) $n^2-25=(n+5)(n-5)$ or "$n^2 = (n-5)(n+5) + 25$" with n = 15, 25, 35.....

3. "If a single digit number a is multiplied by 9 the digit sum is 9" (by inspection). The product is 10(a-1)+10-a i.e. "a-1", "10-a", with digit sum 9. "If a two digit number is multiplied by 9, the digit sum is 9 or 18." (If the number is 10a+b the answer is 100(a-1)+10(9-a+b)+(10-b) with digit sum 18, unless b>a when it is 100a+10(b-a-1)+(10-b) with digit sum 9)
For 3 digit numbers 'abc', the digit sums after multiplication by 9 are 27 (if a≥b and b≥c), 18 (if a<b and b>c, or if a>b and b<c), and 9(if a≤b and b<c).

4(a) 121, 1331, 14641, 161051, 1771561,
10201, 1030301, 104060401,
(see Q.17). The pattern is symmetrical, and like Pascal's Δ up to 11^4, but then is obscured by the "carry" figure. For 101 the pattern is similar and here the carry does not interfere, but the machine capacity is soon exceeded and the less significant digits are lost. The leftmost digits show the Pascal pattern up to 101^9 when the carry figure begins to interfere. 1001^2, 1001^3, 1001^4 etc. also give the pattern, though only the first few numbers of each row are obtained.

5(a) Numbers are 1, 3, 6, 10, 15, 21, 28, 26, 45, 55, 66, 78, 91, 105, 120,
formula $T_n = \frac{n(n+1)}{2}$, with successive differences 1, 2, 3, 4, 5...

6(a) $1 + 3 + 5 + 7 + \ldots + (2n-1) = n^2$ e.g. for n=10, $1 + 3 + \ldots + 19 = 100 = 10^2$
This can be proved by mathematical induction, or by considering the L-shaped regions in (7a); the regions contain 1, 3, 5, 7, ... dots.

(b) Suggests that the sum of two consecutive Δ numbers is a perfect square. The diagram in (7a) gives visual proof that
$T_{n-1} + T_n = n^2$;
this also follows algebraically since $\frac{n(n-1)}{2} + \frac{n(n+1)}{2} = n^2$

7. (b) The sum of the two equal Δ numbers is represented by a dot pattern of a rectangle $n \times (n+1)$. i.e. $T_n = \frac{n(n+1)}{2}$.

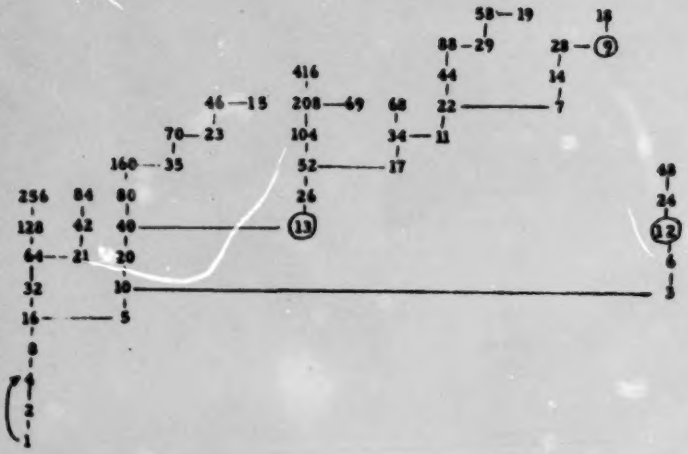
(c) This is rather hard. As a hint, suggest that children examine the differences in (i). If necessary, they can generate more terms of the sequence using (ii).

(i) The sequence 1, 5, 11, 19 has differences 4, 6, 8, 10 and is n^2+3n+1 for $n = 0, 1, 2, 3, \ldots$

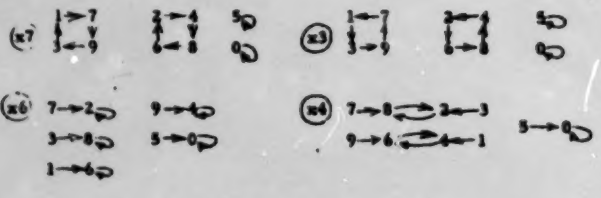
(ii) "The product of 4 consecutive whole numbers is 1 less than a perfect square." $n(n+1)(n+2)(n+3) = (n^2+3n)(n^2+3n+2) = (n^2+3n+1)^2 - 1$.

8. Some of the results, including all starting numbers less than 20, are shown in the "tree". (The tree is shown "topsided" here in a systematic arrangement, where a vertical link indicates an "$x/2$" step and a horizontal link indicates a "$3x + 1$" step) This tree could be constructed as a class activity. Note that once a number is reached which is already part of the tree, the rest of the sequence is known.

It is also possible to extend the tree 'backwards', working away from the "root" at 1. Thus 22 could arise from 44, $(44 \div 2 = 22)$ or from 7, $(7 \times 3 + 1 = 22)$, so going 'outwards', the tree 'branches' here. Pupils may be asked "Which numbers in the chain could arise from two different predecessors?" i.e. where are the "branching points". - The sequences of such numbers is 4, 10, 16, 22, 28, i.e. numbers of the form $6k+4$ (congruent to 4, modulo 6). Vertical branches of the tree which pass (going 'upwards') through a multiple of 3 cannot divide; other vertical branches divide on alternate steps. The chain starting with 27 has 111 steps; other two digit numbers leading to long chains are 31, 41, 54, 55, 62, 63, 71, 73, 82, 91, 94, 95 and 97.



9. The chains obtained are



(see Association of Teachers of Mathematics, Mathematics in Primary Schools, C.U.P., 1967, pages 35-44)

10 (a) $63 = 9 \times 7$, $91 = 7 \times 13$, $221 = 13 \times 17$, $323 = 17 \times 19$, $187 = 11 \times 17$, $247 = 13 \times 19$, $161 = 7 \times 23$, $209 = 11 \times 19$, $119 = 7 \times 17$.

(b) $55 = 5 \times 11$, $105 = 3 \times 5 \times 7$, $231 = 3 \times 7 \times 11$, $51 = 3 \times 17$.

11. "A number is divisible by 3 (or by 9) if, and only if, the sum of its digits is divisible by 3 (or by 9)." The proof is simple, using modulo arithmetic:- Let the number N be 1000a + 100b + 10c + d, (say). Then N = 999a + 99b + 9c + (a+b+c+d) and so N is divisible by 3 or 9 if, and only if, (a+b+c+d) is. (We can express this as "1000a is congruent to a, (mod. 9)" etc., so that a and 1000a have the same remainder on dividing by 9).

12. Primes. The others are 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.
$84=2^2 \times 3 \times 7$, $120=2^3 \times 3 \times 5$, $96=2^5 \times 3$, $180=2^2 \times 3^2 \times 5$, $169=13^2$, $144=2^4 \times 3^2$, $160=2^5 \times 5$.

13. "An odd prime number, N, can be written in the form $x^2 + y^2$ if N is of the form 4k+1, but not otherwise."
The second part of the statement may be proved using congruences ("modulo arithmetic"), modulo 4, as follows. Clearly x and y cannot both be even, or both odd, otherwise N would be even and so not prime. Hence one is even, say y, and the other, x, is odd. Let $x = 2k+1$. Then $x^2 = 4k^2+4k+1$ and so N is congruent to 1 modulo 4 (i.e. leaves remainder 1 on division by 4). Thus it is not possible to have $N=x^2+y^2$ if N is of the form 4k+3.
The first part of the statement is much harder to prove; see Barnard, S. and Child, J.M., Advanced Algebra, Macmillan, 1939, p.186. The result was first proved by Fermat.
A more general result is that every number can be expressed as the sum of 2 squares in at least one way, provided that each of its prime factors of type 4k+3 occurs to an even power in N.
For example $45 = 9 \times 5 = 3^2 \times 5$ and $45 = 6^2+3^2$,
whereas $35 = 5 \times 7$ contains the prime 7 to an odd power, and cannot be so expressed.
The representations of all numbers from 25-100 are given below, where N=(x, y) means $N=x^2+y^2$.

This might form a suitable topic for a class investigation following on the earlier problem.

25→(50) and (4, 3) 26→(5, 1) 29→(5, 2) 32→(4, 4) 34→(5, 3) 36→(6, 0)
37→(6, 1) 40→(6, 2) 41→(5, 4) 45→(6, 3) 49→(7, 0) 50→(7, 1)
52→(6, 4) 53→(7, 2) 58→(7, 3) 61→(6, 5) 64→(8, 0) 65→(8, 1) and (7, 4)
68→(8, 2) 72→(6, 6) 73→(8, 3) 74→(7, 5) 80→(8, 4) 81→(9, 0)
82→(9, 1) 85→(7, 6) and (9, 2) 89→(8, 5) 90→(9, 3) 97→(9, 4)
98→(7, 7) 100→(10, 0) and (8, 6)

14. H. C. F. 's are (15, 40)→5, (35, 42)→7, (49, 77)→7, (48, 49)→1

15. (b, c) L.C.M. 's are (15, 40)→120, (35, 42)→210, (49, 77)→539, (48, 49)→2352.

(d) Product of numbers is 600 1470 3773 2352

The product of H. C. F. and L. C. M. equals the product of the two numbers.

The explanation may be seen by considering <u>factors</u> e. g. for 70 and 16

$70 = 2 \times 5 \times 7$ $16 = 2^3 \times 3 \times 7$ H. C. F. $= 2 \times 7$ L. C. M. $= 2^3 \times 3 \times 5 \times 7$

The product H. C. F. x L. C. M. contains all the factors of both numbers. The L. C. M. contains each factor the greatest number of times it occurs in either number. The H. C. F. contains each factor the least number of times it occurs in either number.

Thus each factor occurs precisely the correct number of times in the product.

16. A very simple exercise (although (f) poses an interesting problem). The aim is to find <u>efficient</u> ways of reaching the results, using knowledge of number facts:-

(a) $12 = 11 + 1$; $100 = 11 + 11 + \ldots\ldots + 1$

(b) $26 = 22 + 2 + 2$; 5 cannot be obtained, since all allowable numbers are even.

(c) $13 = 12 + 1$; $20 = 12 + 2 + 2 + 2 + 2$

Clearly if 1 is allowable all numbers can be obtained in some way with ⊕; likewise in (c) and (e) all numbers are possible, and the interest lies in achieving a solution in only a small number of steps.

Exx. (b) and (d) allow only restricted sets of numbers - in (d) those of the form $2^p \times 3^q$.

(e) $17 = (2 \times 2 \times 2 \times 2) + 1$ (A possible "trick" method:- on some machines, $1 + 2 \times 2$ may give $3 \times 2 = 6$; if this is so, $2 \times 2 \times 2 + 2 \times 2 + 2 + 1$ gives $(8 + 2) \times 2 + 3 = 23$. This may help pupils to become familiar with the operating convention of their machines, but work which is "machine-dependent" should be avoided, on the whole).

(f) Numbers possible with 3 and 5 are

0	5	10	15	20
3	8	13	18	23
6	11	16	21	26
9	14	19	24	29
12	17	22		
15	20	etc.		

Impossible numbers are 1, 2, 4, 7.

All numbers above 7 can be obtained.

For 4 and 7 the possibilities are

0	7	14	21	28
4	11	18	25	32
8	15	22	29	36
12	19	26	33	40
16	23	30	37	
20	27	34	41	
24	31	39		
28	35	etc.		

Impossible numbers are 1, 2, 3, 5, 6, 9, 10, 13, 17.

Since 4 and 6 have H. C. F. 2, no odd number can be expressed as sums of 4 and 6, i.e. only even numbers greater than 2 may be obtained.

The largest unobtainable number for the pair (3, 5) is 7, and for the pair (4, 7) is 17.

We can tabulate results, as follows

a	3	4	2	3	6	2 ...
b	5	7	5	7	9	7 ...
Largest number not expressible as ax+by	7	17	3	11	-	5 ...

In general, for the pair (a, b) the largest unobtainable number is ab-a-b, provided a and b have no common factor. (See Polya, G., Induction and Analogy in Mathematics, Princeton University Press, 1961, p. 73). This could be a further topic for class investigation.

17(a) 4 + 6 = 10, 10 + 10 = 20 etc. Each number is the sum of the two numbers just above it (†). The numbers arise when we consider the number of paths through a rectangular maze with two choices at each vertex, or the number of outcomes when a coin is tossed several times - e.g. there are 4 ways of getting one Head and 3 Tails, and 6 ways of getting two Heads and two Tails when a coin is tossed 4 times - compare the row 1, 4, 6, 4, 1.
161051 arises from 1, 5, 10, 10, 5, 1 in each column of digits, but, because of the 'carry' figures, the '10' becomes '11', and the '5' becomes '6'.

(b) The triangular numbers are in diagonal B - they are the sums of successive sets of numbers, starting at 1, from diagonal A. Similarly the sums of triangular numbers are given by the elements of diagonal B, and sums of the elements of diagonal B by the elements of the next diagonal etc. (These results are easily proved, using the property (†) in 17(a) above).

(c) The sum of each row is twice that of the row above it (and so is 2^n for the nth row) - this follows since each number is 'used twice' in building up the row below it. Pascal's triangle gives the array of binomial coefficients. The binomial theorem enables us to expand expressions of the form $(a + b)^n$, $n = 2, 3, 4, 5, \ldots$.
e.g. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, where the coefficients 1, 3, 3, 1 are those in Pascal's triangle.
It is also used in the theory of probability. Thus the probability of getting exactly two "sixes" in 4 throws of a dice is $6 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2$; the coefficient 6 is from the 1, 4, 6, 4, 1 row of the Δ.

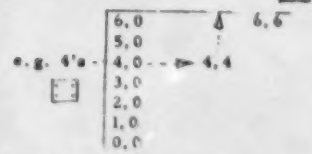
18. In may cases the solutions will be found by trial, but pupils can be asked to suggest systematic ways of solving; some of these are give here.

(a) $(2x + 1) \times 5 = 10x + 5$. Thus 6 becomes 65, 33 becomes 335 } etc. .

(b) If the original number is $10a + b$ the reversed number is $10b + a$. Subtracting we have $9a - 9b$. So $9(a-b) = 45$, and $a-b = 5$. Thus 94, 83, 72, 61 are all possible.

19(a) Since $7 \times 11 \times 13 = 1001$, the number 325 becomes 325325 and, in general, 'xyz' becomes 'xyzxyz'.

(b) Each pattern of a group of pips, say [:.], occurs 8 times (see diagram) .-



Hence the total number of pips is $8(1+2+3+4+5+6) = 8 \times 21 = 168$.

(c) The middle number must be 1/5 of 195
i.e. 39. So the houses are 35, 37, 39, 41, 43.

20.(a) 19998, 29997, 39996, 49995.....
These numbers are 10000-1 multiplied by 2, 3, 4...etc.
i.e. they are 20000-2, 30000-3, etc., so the units digit decreases by 1 at each step and the first digit increases by 1. The sum of digits remains the same, 36.

(b) Jean adds on 10 more than Tommy for each number, so her answer is 100 more.

(c) Try 11 + 6, 22 + 6, 33 + 6,.. 39 satisfies the conditions. The next such number is 39 + 88 (since adding 8 x 11 to the number does not alter the remainder on division by 8 or by 11) i.e. next number is 127. Systematic trial based on the 11-times (or 8-times) table is helpful.

(d) This leads to the equation $x^2 - 36 = 16x$ which has the solutions -2 and 18, of which 18 satisfies both conditions. (It is not to be expected that pupils will solve this by any other method than trial and error - but this problem may make the point that trial, using the calculator, is often a powerful method of solving problems. e.g. to find the positive root of the equation $x^3 - 36 = 16x$, the method of systematic trial is nearly as quick, and is as accurate, as any other. (The root lies between 4 and 5, and between 4.7 and 4.9, and is 4.8411 approximately).

21-25. These exercises may help children whose arithmetic is weak to comprehend such identities as $2 \times (3 \times 5) = (2 \times 3) \times 5$. Without the calculator they may find the mechanics of computation leave them little attention to spare for the behaviour of the answers. Once children have grasped that $6 \times 7 = 7 \times 6$ they need only remember half as many multiplication facts, but until they are fairly competent with number they may be in no position to observe such general regularities. This particular law may easily be demonstrated by examining a rectangular array of dots, say 6×7 - there are 6 rows of 7 or 7 rows of 6. Nevertheless, the

calculator is available to verify that we get the same answer when we multiply two numbers, whatever the order in which we enter them.

Note. It is most important that pupils do not merely enter numbers and operations serially "from left to right", as the expressions are written on the page e.g. 2+3x4. Instead they should work out and record partial answers, for example, the values of bracketed expressions, and so obey the conventions indicated in the way the expression is displayed.

The major points which this work is intended to reinforce may be summed up as follows:-

The inverse nature of $\oplus, \ominus$ and of $\otimes, \oplus$ (Ex 21).

The associative laws for addition $a+(b+c)=(a+b)+c$ (Ex 22) (not for $\ominus$).
and for multiplication $ax(bxc)=(axb)xc$ (Ex 23 not for $\oplus$).

The commutative laws for addition $a+b=b+a$ (Ex 22) (not for $\ominus$).
and for multiplication $axb=bxa$ (Ex 23) (not for $\oplus$).

The distributive laws $a(b+c)=ab+ac$, $a(b-c)=ab-ac$,
(Ex 24 and 25) $(b+c)a=ba+ca$, $(b-c)a=ba-ca$;
note that $\oplus$ is distributive only on the right, $(a \pm b) \div c = a/c \pm b/c$

This work is more difficult than it seems — it is perhaps not surprising that pupils often have some problems here. An interesting exercise for the teacher (and perhaps for a few of the abler pupils) is to examine the 16 possible expressions obtainable from $6 \square (4 \triangle 2)$ when $\{+, -, \times, \div\}$ are substituted for $\square$ and $\triangle$ independently in turn - to determine in which cases a general statement can be made. The expression $(6 \square 4) \triangle 2$ may be treated similarly.

The last part of Q.15 may be helpful in showing the principle of long multiplication. If notions of place value are limited, the usual algorithm may be just a ritual which (sometimes!) gives the right answer. Using the calculator, as here, we can show how the component parts arise - i.e. use the calculator to increase understanding and, perhaps, proficiency in doing calculations without the calculator.

26. If we write the expression as $3 \overset{a}{+} 4 \overset{b}{\times} 2 \overset{c}{+} 7$, inserting brackets directs us to perform the operations a, b, c in some particular order. The possible orders, with the corresponding results, are:-

(abc)	21	(bac)	18	(cab)	63
(acb)	63	(bca)	18	(cba)	39

For $2 \overset{a}{\times} 3 \overset{b}{+} 4 \overset{c}{+} 1$ we have

(abc)11, (acb)11, (bac)15, (bca)16, (cab)11, (cba)16.

For $9 \overset{a}{-} 4 \overset{b}{-} 1 \overset{c}{+} 3$ we have

(abc)7, (acb)1, (bac)9, (bca)3, (cab)1, (cba)9,

and for $12 \overset{a}{\times} 5 \overset{b}{-} 1 \overset{c}{+} 3$ we have

(abc)62, (acb)56, (bac)51, (bca)8, (cab)56, (cba)12.

If in the classroom, there are present calculators with different operating conventions, it is worth encouraging an examination of the differences between them (at an appropriate stage, when children's understanding of the conventions is sufficient to cope with such a situation). There is an analogy with other common pieces of equipment (radios, tape-recorders, washing machines, motor cars...) which despite minor differences in their controls, from type to type, perform similar functions. Pupils should at first, or if in doubt, always experiment with small whole number problems to check the functioning of an unfamiliar machine.

Exploring Numbers - number patterns and investigations: (Series A)

Pupils' Activities

1. Your calculator can help you to examine number patterns. Here are some examples. (Try to guess each answer before you work it out on the calculator)

(a)	(b)	(c)
1 x 2 =	0 + 2 =	1 + 2 =
2 x 2 =	2 + 2 =	3 + 2 =
3 x 2 =	4 + 2 =	5 + 2 =
.....	6 + 2 =	7 + 2 =
	.....	.....

The dots mean "try to continue the pattern"

DO NOT WRITE ON THIS SHEET - put your answers on a separate page

(d) How would you continue each of the patterns?

3, 6, 9, 12, ...

1, 4, 7, 10, 13, ...

2, 5, 8, 11, 14, ...

1, 3, 6, 10, 21, ... [Notice $1 \xrightarrow{\text{add } 2} 3 \xrightarrow{\text{add } 3} 6 \xrightarrow{\text{add } 4} 10 \ldots$]

1, 4, 9, 16, 25, ... [Notice $1 \xrightarrow{\text{add } 3} 4 \xrightarrow{\text{add } 5} 9 \xrightarrow{\text{add } 7} 16 \ldots$]

9, 18, 27, 36, 45, .. [Notice $1 + 8 = 9,\ 2 + 7 = 9,\ 3 + 6 = 9 \ldots$]

---

3. Work out 3 x 9 = 27 (Notice that 2 + 7 = 9)
and 4 x 9 = 36 (Notice that 3 + 6 = 9).

Do the same with 5 x 9,

6 x 9... and so on.

How far does the pattern continue?

(Don't stop at 10 x 9 - go on to 11, 12, ... 20, 21, 22, ...

....11 x 9, 12 x 9, ..., 20 x 9, 21 x 9, 22 x 9, ... and so on).

Try some numbers over 100 - like 125 x 9, 126 x 9, ...

CHECK. (with the calculator)

Copyright © F.R. Watson, University of Keele, 1979

Note. The reproduction of Pupil material by teachers for use with their own pupils is expressly permitted.

2. In some of the examples you may not <u>need</u> to use the calculators - but it is a good idea to check your work afterwards with the calculator. To suggest this, we shall write CHECK, meaning "check with the calculator".

(a) Work out each of these pairs and compare the answers in each column.

1 x 1 =	compared with 2 x 0 =
2 x 2 =	compared with 3 x 1 =
3 x 3 =	..... 4 x 2 =
4 x 4 =	..... 5 x 3 =
.....	.....

Continue the pattern. Try it with some bigger numbers like 29 x 29, using the calculator.

(b) Work out 10 x 20 and compare with 15 x 15

20 x 30	25 x 25
30 x 40	35 x 35
.....	.....

Continue the pattern. Try some big numbers like 170 x 180 with the calculator.

4. <u>Write down</u> your answers at each stage in the following questions:-

(a) Work out (i) 11 x 11, (ii) 11 x 11 x 11 = 121 x 11, (iii) 11 x 11 x 11 x 11, (iv) 11 x 11 x 11 x 11 x 11, and (v) 11 x 11 x 11 x 11 x 11 x 11.

Use each answer to help you to find the next one.

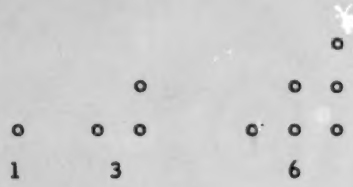
(No. (v) has 6 "elevens" and is sometimes written as 11^6). CHECK.

(b) Work out and write down,

(i) 101 x 101, (ii) 101 x 101 x 101, (iii) 101 x 101 x 101 x 101. CHECK.

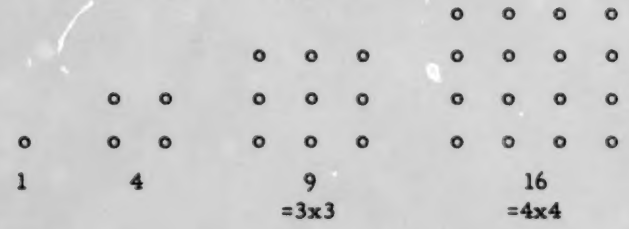
Do you notice anything about the patterns of digits in (a) and (b)?

5. The numbers 1, 3, 6, 10, 15, 21, ... are sometimes called <u>triangular</u> numbers. This is because they can be arranged in triangular patterns like this:



(a) Draw the diagrams for 10 and 15.
Write down the next two Δ numbers after 21.
Can you see how to get each Δ number from the one before?
Find all the Δ numbers less than 100 (but don't draw the diagrams).

(b) The <u>square</u> numbers also form a pattern

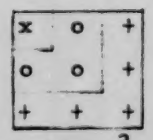


Work out the next two square numbers and draw the diagrams for them.
Find all the square numbers up to 400. CHECK results with the calculator.

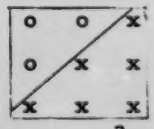
---

7. (a) These diagrams can help you to explain why the results you found in 6 are true.

Can you see how?



$1 + 3 + 5 = 3^2 = 9$

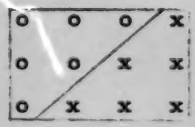


$3 + 6 = 3^2 = 9$

$\Delta + \Delta = \square$

(b) Work out 1 x 2, 2 x 3, 3 x 4, 4 x 5, ...
and compare the answers with the triangular numbers.
Can you explain?

<u>Hint</u>



(c) (i) Can you continue this pattern? 1. 5. 11, 19, 29, 41.
If you can't see the rule, part (ii) may help you

(ii) Continue this pattern $(1 \times 2 \times 3 \times 4) + 1 = 25 = 5 \times 5$
$(2 \times 3 \times 4 \times 5) + 1 = 121 = 11 \times 11$
$(3 \times 4 \times 5 \times 6) + 1 =$ .....
.....

Compare with the numbers in part (i).

6. (a) Work out $1 + 3 + 5$

$1 + 3 + 5 + 7$

$1 + 3 + 5 + 7 + 9$ Do you notice anything?

Make a guess about $1 + 3 + 5 + 7 + 9 + 11$. Now check your guess.
Were you right? CHECK.
Try this pattern for some bigger numbers e.g. $1 + 3 + 5 + \ldots + 17 + 19$
Make up some more examples of your own to test whether it always works.

(b) 6 and 10 are consecutive ('next to each other') triangular numbers.
So are 15 and 21. Notice that $6 + 10 = 16$, a square number, and
$15 + 21 = 36$, another square number. Does this always work?
Try some of the Δ numbers you found in 5(a).

---

8. An arrow game

Rules. "Start with any number. If it is even, then halve it. If it is odd, treble it and add one. Then follow the same instructions with the new number. Draw a chain to show the list of numbers you get."

Here is an example. Suppose you start with 13 :-
13 is odd, so treble it and add one. (Next number is $13 \times 3 + 1 = 40$).
40 is even, so halve it, to give 20.
20 is even, so the next number is 10; then 10 becomes 5.
5 gives 16 (because you treble it and add 1), then you get 8, then 4, then 2, then 1.
1 gives 4, then 2, then 1 again.
So the chain is (13) → 40 → 20 → 10 → 5 → 16 → 8 → 4 → 2 → 1

Now try another starting number, say (12) or (9).

This arrow chain always seems to end up at 1 eventually. Try some other starting numbers and see if this is so.
27 has a very long chain, but does come down to 1 eventually. (No-one has ever proved that this will happen for every number, so maybe you will find one that doesn't!)

Copyright © F.R. Watson, University of Keele, 1979

9. (a) Work out each of these products:-

19	29	79
x7	x7	x7

What is the units digit in each answer?

What would you write in the box to make each of these sentences correct?:-

'If the units digit of a number is 9, multiplying it by 7 changes the units digit to ☐.'

'If the units digit of a number is 3, multiplying it by 7 changes the units digit to ☐.'

(b) Now copy and complete this table:-

Units digit (at first)	9	3	1	7	2	4	6	8	5	0
Units digit (after multiplication by 7)	3	1								

'contd...'

---

10. Factors

(a) 72 = 8 x 9. We say 8 and 9 are factors of 72 - that is, they each divide exactly into 72.

Try to split up these numbers into two factors, in a similar way:-
63, 91, 221, 323, 187, 247, 161, 209, 119.

(b) Example 'Find factors of 98'.

Since 98 = 2 x 49, both 2 and 49 are factors of 98.
Since 49 = 7 x 7, we can write 98 = 2 x 7 x 7.
Notice that neither of the factors 2 and 7 will 'split up' any more - we call them prime numbers. We have split up 98 into a product of prime factors.

In the same way 273 = 3 x 91. Use your calculator to find a number which will divide exactly into 91 and so split 273 into a product of 3 prime factors.

Split up these numbers into prime factors:- 55, 105, 231, 51.

(c) We can write 98 = 2 x 49 = 7 x 14; we can see that 2, 7, 14, and 49 are all factors of 98 - the complete list of factors of 98 is 1, 2, 7, 14, 49, 98.

Find all the possible pairs of factors of each of these numbers
9, 15, 25, 18, 100.
In each case, write a complete list of the factors of the number.
(e.g. for 100 the list is 1, 2, 4, 5, 10, 20, 25, 50, 100 – there are no others, since no other number will divide exactly into 100).

9. (c) We can make an arrow chain to illustrate part of the table in (b) where the arrow means "multiply by 7". (This means 4 changes into 8, 8 changes into 6, and so on. The last digit, and the number it changes into when you multiply by 7, are joined by the arrow)

Can you make a chain of arrows, like this, for the numbers 1, 3, 7, 9?

(d) We have tried multiplication by 7.
Repeat the steps of (a), (b), (c) in the first part of this question but this time using multiplication by 3.
Can you make arrow chains in a similar way?

(e) Investigate multiplication by 6 in the same way. Some of the chain patterns you get are

See if you can find the others.

(f) Investigate multiplication by 4 in the same way.
What chain patterns can you find?

---

11. Rules for divisibility (We can use these to tell when one number can be divided exactly by another)

(a) Do you know a quick way of deciding whether a number will divide exactly by 2? Try these:- 43, 26, 91, 72, 88, 99.
Can you see the rule?

(b) Here is a list of numbers. Find which of them are divisible by 3 (that is, will divide exactly by 3).
51, 27, 17, 15, 58, 63, 83, 126, 96
Which of the numbers in the list are divisible by 9?

(c) Add up the digits in each number in (b):-
(e.g. sum of digits of 51 is 5 + 1 = 6, of 27 is 2 + 7 = 9).

Make a table:-

Number	Sum of digits	Divisible by 3?	Divisible by 9?
51	6	yes	no
27	9	yes	yes
.....	.....	.....	..... etc.

Can you see any connection between divisibility by 3 and the sum of the digits? Try also to find a rule for divisibility by 9. Try some more numbers to see if your rule works.

Copyright © E.R. Watson, University of Keele, 1979

12. Some numbers don't have any divisors (factors) except themselves and 1. An example is 17 - the only numbers which will divide exactly into 17 are 1 and 17 itself. Numbers like this are called <u>prime numbers</u>.

Try to find a divisor of 23. Is 23 a prime number?

The first few prime numbers are 2, 3, 5, 7, 11, 13, 17, ...
There are 18 others less than 100.
* ? can you find them?

Split each of the following numbers into a product of prime factors like this:- 90 = 2 x 3 x 3 x 5 (since 90 = 9 x 10 = 3 x 3 x 2 x 5).

84, 120, 96, 180, 169, 144, 160.

---

14. <u>Common factors</u>

The set of factors of 35 is {1, 5, 7 and 35} ; (35 = 5 x 7).
The set of factors of 45 is {1, 3, 5, 9, 5, 45}, (45 = 3 x 3 x 5).
The number 5 is a factor of both 35 and 45.
We say 5 is a <u>common</u> factor of 35 and 45 (common means 'belonging to both').

Find common factors of 72 and 32
of 21 and 35
of 12 and 20

The common factors of 56 and 40 are 1, 2, 4, and 8.
The largest of these is 8, so 8 is the <u>highest common factor</u> of 56 and 40.

Find the highest common factor of 15 and 40
of 35 and 42
of 49 and 77
of 48 and 49

(Write down your answer for use later on in Q.15)

Copyright © F.R. Watson, University of Keele, 1979

13. In question 5 we investigated square numbers 1, 4, 9, 16, 25, 36...
Some numbers can be formed by adding up <u>two</u> square numbers.
e.g. 20 = 4 + 16, and 13 = 4 + 9
But 11 <u>cannot</u> be formed in this way. (Try it, and see)
Here is a list ('set') of numbers . (They are all <u>prime</u> numbers) Find out which of these can be made up by adding up two square numbers.

$\{3,\ 5,\ 7,\ 17,\ 19,\ 29,\ 37,\ 41,\ 53,\ 59,\ 61\}$

Is there anything special about the ones which <u>cannot</u> be written as the sum of two square numbers? <u>Hint</u>. Try dividing each of the numbers in the list by 4. If you can't see the rule, test a few more prime numbers like 67, 71, 73, 79, 83.

---

15. <u>Multiples</u>

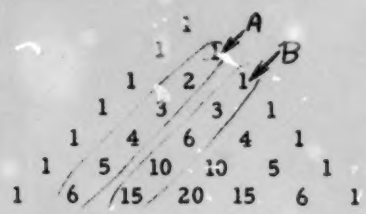
(a) The multiples of 15 are 15, 30, 45, 60, 75, 90.....150, 165,.....
Can you find a multiple of 15 which is <u>also</u> a multiple of 40?
This is a <u>common</u> multiple of 15 and 40 ('common' means "shared by both")
What is the <u>least</u> number which is a multiple of 15 and also a multiple of 40?
(This is the <u>least</u> <u>common</u> multiple (L. C. M.) of 15 and 40)

(b) Find a common multiple of 35 and 42. Find their <u>least</u> common multiple (L. C. M.).

(c) Find the L. C. M. of 49 and 77, and the L. C. M. of 48 and 49.

(d) Here are some pairs of numbers:- (15 and 40); (35 and 42); (49 and 77); (48 and 49).
For each of these pairs of numbers you have worked out the highest common factor (H. C. F.) and the least common multiple (L. C. M.).
Now, for each pair, work out the product (H. C. F.) x (L. C. M.) and compare it with the product of the two numbers. What do you notice?
Try this with some other pairs of numbers which you choose yourself. Find the H. C. F. and L. C. M. for each pair.

17. **The answers in question 4 should follow the pattern of this table which is called Pascal's triangle. It contains some interesting patterns.**



(a) Can you see the rule of this table? (Hint: 4 + 6 = ?)
Can you write the next two lines?
Can you see why you got 1 6 1 0 5 1 in 2(a)?

(b) Can you see the triangular numbers in this table?
In diagonal line A notice that 1 + 2 + 3 = 6
and that 1 + 2 + 3 + 4 = 10
In diagonal line B notice that 1 + 3 + 6 + 10 = 20
Can you see any other similar patterns in this table? Hint. 1 + 3 + 6 = 10 etc.

(c) Add the numbers in each of the rows of the table:-

1 + 1 = 2
1 + 2 + 1 = 4
1 + 3 + 3 + 1 =
etc.

What do you notice? Can you explain this?

---

16. Challenge

(a) Suppose you are allowed to press ONLY the keys (1), (+) and (=) on your calculator.
You can press the keys as often as you like, and in any order.
How would you get 5 in the display?
How would you get 12?
How would you get 15?
How would you get 100?
} Try to do each one in the easiest way.

(b) Suppose you are allowed to press ONLY the keys (2), (+) and (=) as often as you like and in any order.
How would you get each of these numbers? 8, 22, 26
Can you get the number 5?

(c) You are allowed to press ONLY the keys (1), (2), (+) and (=), in any order and as many times as you wish. Can you get these numbers? 5, 13, 23, 9, 20.
Make up some more questions of your own, using only the (+), and (=) keys with some numbers.

(contd....)

Copyright © F.R. Watson, University of Keele, 1979

Puzzles

(18) Some puzzles, which your calculator may help you to solve.

(a) Choose any one of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9.
Multiply the number you chose by 2 and add 1. Now multiply your answer by 5. Try this again, choosing a different number. What do you notice?
Try some more numbers to see if your idea works every time.
(If at first you don't notice anything, try a few more numbers - but read the instructions carefully to make sure you are following them correctly).
Try the same thing, choosing a number between 10 and 100.
Try this trick on your friends (You can pretend to be a mind-reader!)
* Can you see why this trick always works?
** Make up another trick like this one.

(b) A certain number has two digits, and when the number obtained by interchanging these two digits is subtracted from it, the result is 45. What was the original number? (Is there more than one answer? How many different answers can you find?)

---

16. contd.

(d) You are allowed to press the keys ②, ③, ⊗ and = ONLY, in any order and as many times as you wish. See if you can get these numbers:- 6, 9, 8, 12.

(e) You are allowed to press the keys ①, ②, +, ⊗ and = in any order and as often as you wish. Can you get these numbers?
17, 23, 33.
Now try again, but this time don't use numbers like 22, 12 and so on, where you press two number keys one after the other.
Make up some more questions like these (a) allowing only the use of ⊗, = and numbers (b) allowing both + and ⊗ as well as = and numbers.

(f) If you are allowed to press ONLY ③, ⑤, + and =, you can get 8 by 3 + 5 and 12 by 3 + 3 + 3 + 3. Find which of the numbers from 1 to 20 you can get in this way. What is the largest number you cannot get? (It is possible to get every number over 15)
Investigate the same problem (i) with 3 and 5 replaced by 4 and 7 (all numbers over 28 are possible), (ii) replacing 3 and 5 by 4 and 6.
Try several other pairs of numbers; in each case find what numbers you cannot get with these rules.
Try these:- 2 and 5, 3 and 7, 6 and 9, 2 and 7.
Can you see any pattern? If you are given the two numbers, can you see a quick way of deciding which is the largest number which cannot be obtained by following these rules.

Puzzles (cont'd)

(19) (a) Take any number of three digits (that is, a number between 100 and 1000) Multiply it by 7, multiply the answer by 11, and multiply that answer by 13. What happens? Try another 3 digit number.

** See if you can find out why this happens.

(b) How many pips are there altogether on a complete set of dominoes?

(The biggest domino in a set is usually marked [6|6]

The others are [6|0] ..... [6|5]

then [5|0] ..... [5|5]

[4|0] ..... [4|4]

and so on....)

** Can you find an easier way than just counting them all?

(c) Five houses in a street are next to each other in a row. Their numbers, which are all odd, add up to 195. What is the lowest number?

---

21. (a) Work out each of the following. What do you notice? Try to explain what is happening. Afterwards talk to your teacher about your work.

7 + 8 = (15)	7 x 8 =
15 - 8 = (7)	56 ÷ 8 =
10 + 6 =	6 x 9 =
16 - 6 =	54 ÷ 9 =
123 + 17 =	100 ÷ 5 =
140 - 17 =	20 x 5 =
24 - 8 =	CHECK each one with your calculator.
16 + 8 =	

We say that addition and subtraction are inverse operations; this means that one "undoes the effect of the other". (†)
In the same way, if we multiply a number by 4 and then divide the answer by 4, we shall get back to the original number, since the division "undoes" the effect of the multiplication.
Multiplication and division are another pair of 'inverse operations'. (cont'd...)

(†) Adding a number on, and then taking it off again (subtracting it) doesn't change the result.

Copyright © F.R. Watson, University of Keele, 1979

Puzzles, (contd.) ..

(20) (a) Work out 9999 x 2
9999 x 3
9999 x 4
and so on...

What do you notice?
* Can you explain why?

(b) Tommy adds up all the numbers from 1 to 10. Jean adds up all the numbers from 11 to 20. Jean's total is more than Tommy's. How much more?

* Can you find a quick way of doing this?

(c) Find a number which is 7 greater than a multiple of 8 and 5 less than a multiple of 11. (There is one less than 50).

* What is the next highest number which obeys these rules?

(d) Try to find a number which obeys the following rules. (Experiment until you find one which works)

i) It must be bigger than 8

ii) If you multiply the number by itself and then subtract 36 you get the same answer as if you had multiplied the original number by 16.

---

21. contd.

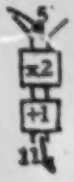
(b) Some more examples on <u>inverse</u> operations:

Work out the following. Try to <u>predict</u> answers before they appear.

113 - 5	667 ÷ 29
108 + 5	23 x 29
237 - 129	312 ÷ 24
108 + 129	13 x 24
63 ÷ 7	1247 ÷ 43
9 x 7	29 x 43

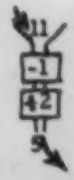
Make up some more examples of your own. Try them out, then show your work to your teacher.

(c) We can invent a "function machine" like this:-
Here is how it works. You put in a number, say 5.
First it is doubled, then 1 is added.
The answer which comes out, is 11.



Machine A

The second machine, B, reverses the effect of the first. If you put in 11, 5 will come out.
Try putting a different number into the first 'machine' A.
Then put the result into the second machine B.
Try this with other numbers. Does it always work?



Machine B

** Invent two other function machines like these, so that one reverses the effect of the other. Test them with some numbers as you did for A and B.

22. "How brackets work." Part I: Addition

Note. Have ready a piece of scrap paper and a pencil so you can write down some answers as you find them (your calculator may have a memory which you could use instead for this).

Work out $2 + 3 =$

$(2 + 3) + 4 =$

$3 + 4 =$

$2 + (3 + 4) =$

} Write down each answer on paper

Do you think that $2 + (3 + 4) = (2 + 3) + 4$?

Try these

$(3 + 4) + 11 =$

$7 + 11 =$

$3 + (7 + 11) =$

} Write down your answers

$(248 + 351) + 192 =$

$351 + 192 =$

$248 + (351 + 192) =$

} Write down your answers

Write out some more examples like these and CHECK them.
Can you explain what you have found, using these three numbers as an example?
4 + 6 + 2.
We can use this rule to help us when we are adding without a calculator.
Example. $7 + 6 = 7 + (3 + 3) = (7 + 3) + 3 = 10 + 3 = 13$

(Contd...)

23. "How brackets work." Part II: Multiplication

Work these out

$2 \times 3 =$
$6 \times 5 =$
that is $(2 \times 3) \times 5 =$

(Write down each answer)

$3 \times 5 =$
$2 \times 15 =$
that is $2 \times (3 \times 5) =$

Compare these two

$40 \times 3 =$
that is $(5 \times 8) \times 3 =$

(Write down each answer)

$8 \times 3 =$
$5 \times 24 =$
that is $5 \times (8 \times 3) =$

Compare these

You will need the calculator for these:-

$27 \times 23 =$
$(27 \times 23) \times 19 =$
$621 \times 19 =$
$23 \times 19 =$

$27 \times (437) =$
that is $27 \times (23 \times 19) =$

Compare

$36 \times 5 =$
$180 \times 75 =$
$(36 \times 5) \times 75 =$

$5 \times 75 =$
$36 \times 375 =$
$36 \times (5 \times 75) =$

Compare

Write out some more examples like these and CHECK them with your calculator.
*See if you can write down a rule, in your own words, about multiplying a row of numbers, like $5 \times 2 \times 4$.
We can use this rule to help us in multiplying without a calculator
e.g. $5 \times 4 \times 6 = 5 \times (2 \times 2) \times 6 = (5 \times 2) \times (2 \times 6) = 10 \times 12 = 120$
(or you might say $5 \times 4 \times 6 = 20 \times 6 = 10 \times 2 \times 6 = 10 \times 12 = 120$).

(contd...)

22. contd. "Which way round?" (adding)

Some of these statements are wrong. Which ones?
Look at each one first to decide, and then CHECK using your calculator.

$1 + 2 \stackrel{?}{=} 2 + 1$

$11 + 7 \stackrel{?}{=} 7 + 11$

$11 + 7 + 8 \stackrel{?}{=} 7 + 11 + 8$

$11 - 7 + 20 \stackrel{?}{=} 7 - 11 + 20$

$11 - 5 + 8 \stackrel{?}{=} 5 - 11 + 8$

$10 + 5 - 2 \stackrel{?}{=} 10 + 2 - 5$

$40 + 8 - 3 \stackrel{?}{=} 40 + 3 - 8$

$1 + 2 + 3 \stackrel{?}{=} 1 + 3 + 2$

$1 - 2 + 3 \stackrel{?}{=} 1 - 3 + 2$

These examples illustrate the following rules:-

How brackets work - adding

A) When adding a row of numbers we can group them together as we wish. If we write a, b, c to represent the 3 numbers then

$$\boxed{a + (b + c) = (a + b) + c}$$

Which way round? - adding

B) If a and b are any two numbers then $\boxed{a + b = b + a}$
but $\boxed{a - b}$ is not the same as $\boxed{b - a}$

(A) is sometimes called the Associative Law; (B) is sometimes called the Commutative Law. *There is only one exception to the statement in rule (B) about subtraction. Can you find two numbers a and b for which $(a - b)$ is equal to $(b - a)$?

---

23. contd. "Which way round?" (multiplication)

Try to decide which of these statements are correct and which are wrong
Then CHECK with your calculator.

$12 \times 3 \stackrel{?}{=} 3 \times 12$

$48 \times 2 \stackrel{?}{=} 2 \times 48$

$11 \times 3 \stackrel{?}{=} 3 \times 11$

$(8 \div 2) \times 100 \stackrel{?}{=} (2 \div 8) \times 100$

$(2 \times 3) \times 100 \stackrel{?}{=} (3 \times 2) \times 100$

$12 \div 3 \stackrel{?}{=} 3 \div 12$

$48 \div 6 \stackrel{?}{=} 6 \div 48$

$22 \div 11 \stackrel{?}{=} 11 \div 22$

$(12 \div 6) \times 20 \stackrel{?}{=} (6 \div 12) \times 20$

$(4 \times 7) \times 10 \stackrel{?}{=} (7 \times 4) \times 10$

These examples illustrate the following rules

(A) When multiplying a row of numbers we can group them together as we wish. If we write a, b, c to represent the three numbers then

$$\boxed{a \times (b \times c) = (a \times b) \times c}$$

(This is the Associative Law for multiplication)

(B) If a and b are any two numbers then $\boxed{a \times b = b \times a}$ (This is the Commutative Law for multiplication), but $\boxed{a \div b}$ is not the same as $\boxed{b \div a}$

* There is just one exception to the part of rule (B) involving division. Can you find two numbers a and b for which $(a \div b)$ is equal to $(b \div a)$?

Copyright © F.R. Watson, University of Keele, 1979

24. "How brackets work." Part III: Addition with multiplication

What is the answer when you work out 3 x (4 + 5)?
If your answer is under 20, you are wrong! (Did you remember to work out the bracket first?)
Note. We use brackets in a calculation to tell us the order in which we do the various steps. Thus, 3 x (4 + 5) means "add 4 and 5 and multiply the result by 3" - so the answer is 27. If we had wanted to show that the multiplication should be done before the addition we could have written (3 x 4) + 5. Then the answer would have been 12 + 5, which is 17.

Some of the statements which follow are correct, others are wrong. (Remember to work out the brackets first).
Decide which are correct, and afterwards CHECK them all.
As you work out each bracket or part of an example, write down your working with pencil and paper.

$2 \times (4 + 6) \stackrel{?}{=} (2 \times 4) + (2 \times 6)$

$(7 \times 2) + (7 \times 3) \stackrel{?}{=} 7 \times (2 + 3)$

$(9 \times 11) - (9 \times 2) \stackrel{?}{=} 9 \times (11 - 2)$

$6 + (4 \times 2) \stackrel{?}{=} (6 + 4) \times (6 + 2)$

$5 \times (8 - 2) \stackrel{?}{=} (5 \times 8) - (5 \times 2)$

$(3 \times 3) + (3 \times 2) \stackrel{?}{=} 3 \times (3 + 2)$

$24 \div (6 + 2) \stackrel{?}{=} (24 \div \quad) + (24 \div 2)$

$(6 \times 9) - (6 \times 4) \stackrel{?}{=} 6 \times (9 - 4)$

$2 \times (7 + 3) \stackrel{?}{=} (2 \times 7) + (2 \times \quad)$

$(11 \times 15) + (11 \times 3) \stackrel{?}{=} 11 \times (15 + \quad)$

(cont...)

25. "How brackets work." Part IV: More on the Distributive Law

Decide which of the following statements are correct and which are wrong. Afterwards CHECK. Don't forget to write down the answer for each expression written in brackets.

$(4 + 6) \times 2 \stackrel{?}{=} (4 \times 2) + (6 \times 2)$

$(2 \times 7) + (3 \times 7) \stackrel{?}{=} (2 - 3) \times 7$

$(9 - 5) \times 3 \stackrel{?}{=} (9 \times 3) - (5 \times 3)$

$(7 - 3) \times 2 \stackrel{?}{=} (7 \times 2) - (3 \times 2)$

$(4 + 6) \div 2 \stackrel{?}{=} 4/2 + 6/2$

$(15 + 9) \div 3 \stackrel{?}{=} 15/3 + 9/3$

$100/5 - 25/5 \stackrel{?}{=} (100 - 25) \div 5$

$(2 + 4) \times 7 \stackrel{?}{=} (2 \times 7) + (4 \times 7)$

$(3 \times 3) + (2 \times 3) \stackrel{?}{=} (3 + 2) \times 3$

$(8 \times 5) - (2 \times 5) \stackrel{?}{=} (8 - 2) \times 5$

$(9 \times 6) - (4 \times 6) \stackrel{?}{=} (9 - 4) \times 6$

NOTE. 6/2 means "6 ÷ 2" (that is, "3").

$(15 - 9) \div 3 \stackrel{?}{=} 15/3 - 9/3$

$35/7 - 21/7 \stackrel{?}{=} (35 - 21) \div 7$

These all illustrate another distributive law

If a, b, c are three numbers then $\boxed{(b + c) \times a = b \times a + c \times a}$

and $\boxed{(b - c) \times a = b \times a - c \times a}$

This time the rule does work for division too

$\boxed{(b+c) \div a = b/c + c/a}$ and $\boxed{(b-c) \div a = b/a - c/a}$

But notice that it only works when the division sign follows the bracket

$30 \div (3 + 2)$ is 6 but $30/3 + 30/2$ is 25.

(cont...)

Copyright © F. R. Watson, University of Keele, 1979

24. (contd.)

You may have discovered that these examples all illustrate two general rules:

If a, b, c are three numbers then $\boxed{a \times (b + c) = (a \times b) + (a \times c)}$

and $\boxed{a \times (b - c) = (a \times b) - (a \times c)}$

This rule is sometimes called the Distributive Law.
Notice that there is a multiplication sign ⊠ outside the bracket and either an addition sign ⊞ or a subtraction sign ⊟ inside the bracket.
Note that the rule doesn't work for division.

Note that you cannot interchange ⊞ and ⊠ in the Distributive Law, because $a + (b \times c)$ and $(a + b) \times (a + c)$ are different.

Try these:- $7 + (3 \times 2)$ compared with $(7 + 3) \times (7 + 2)$

$3 + (5 \times 4)$ compared with $(3 + 5) \times (3 + 4)$

Make up some more examples where the Law is obeyed, and CHECK them.

25. (contd.)

Do you know how to do 'long multiplication'?

To work out a product like 327 x 43 you write

```
  327
   43
-----
13080  ← This is 40 x 327
  981  ← This is 3 x 327
-----
14061
```

The last line is found by adding 40 x 327 and 3 x 327 to give (40 + 3) x 327, that is 43 x 327.

CHECK with your calculator 40 x 327 =
3 x 327 =
43 x 327 =

Now do the same for 107 x 29:-
First do the 'long multiplication',

```
  107
   29
-----
  □   ← 20 x 107
  □   ← 9 x 107
-----
      ← 29 x 107
```

and then CHECK as above.

Make up another multiplication problem of your own.
*Afterwards you might like to try 274 x 143.

26. (Hard). Do this exercise without your calculator. Then talk to your teacher afterwards.

Why do we always drive cars on the left-hand side of the road? In some countries cars are driven on the right-hand side; it doesn't matter which rule we use as long as we all agree to do the same! If you are asked to work out 2 + 3 x 7, you ought to get 23 (since 3 x 7 = 21) and 2 + 21 = 23). By convention (custom) we always do multiplication before addition - if everyone does it the same way there is no confusion. Sidney likes to be different - he does the multiplication after the addition, so he says 2 + 3 = 5, 5 x 7 = 35. Sidney is wrong, because everyone else has agreed that multiplication comes first. If we want to do the addition first, we put in some brackets like this (2 + 3) x 7; now the correct answer is 35.

Here is a calculation

3 + 4 x 2 + 5 = ?

If we put brackets in like this, 3 + (4 x 2) + 5, we get 16 which is the usual answer, since multiplication should be done first. But if we put brackets in like this, (3 + 4) x 2 + 5, the correct answer becomes 19. Brackets can be put in in several other ways e.g. 3 + 4 x (2 + 5) See how many different ways you can find, and how many different "correct" answers you can get. (Sidney can get four altogether).

26. (contd)

Do the same with these calculations as you did with 3 + 4 x 2 + 5. Put in brackets in as many different ways as you can think of, and see how many different answers you can get.

(i) 2 x 3 + 4 + 1

(ii) 9 - 4 - 1 + 3

(iii) 12 x 5 - 1 + 3

(Sidney can get 4 different answers in no. (i), 4 in no. (ii), and 5 in no. (iii)!).

Note. If there are no brackets in the expression there is only one correct answer. The rule or convention is that we do multiplication before addition and subtraction, and addition and subtraction in order from left to right. The other answers would be correct if the brackets had been put in.

Note. (talk to your teacher about this):-
On some calculators, if you press the keys in order 2 + 3 x 7 you will get 35. This is because these calculators use a different convention - they carry out instructions in the order in which you press the keys; 2 + 3 = 5, then 5 x 7 = 35. This can be confusing and leads to mistakes, so that many of the newer calculators have now changed over to the same convention as we use.

© Copyright, F.R. Watson, 1979

Exploring Numbers - Number Patterns and Investigations

Pupil's Activities. (Series A)

(Some suggestions for using the electronic calculator).

(A set of Teachers' notes and solutions is available)

Copyright © F.R. Watson, University of Keele, 1979

Note. The reproduction of Pupil material by teachers for use with their own pupils is expressly permitted.

EXPLORING NUMBERS

Some Investigations with a Calculator

Series B

TEACHERS' NOTES

F.R. Watson
Institute of Education
University of Keele

Copyright © F.R. Watson, University of Keele, 1979

Note. The reproduction of Pupil material by teachers for use with their own pupils is expressly permitted.

The advent of the 'pocket' calculator seems likely to have a profound effect on mathematics teaching before long. One way in which it can be used immediately, without any change in 'syllabus', is as a means of exploration and investigation of number properties and patterns.

It is hoped these suggestions will be a useful starting point for teachers, who will modify and expand them to suit their own circumstances. I should welcome any comments, and notes of errors which, perhaps inevitably, will have escaped my attention.

Several people have helped in the production of these materials, but I should particularly like to thank Mrs. B. Wiggins and Mrs. C. Lea for their contributions in typing and printing respectively.

F.R. Watson
February 1979

Number Patterns and Investigations (Series B; for 13-16 years)

Introduction

These activities are based on two beliefs - that most pupils can be led to an interest in the properties of numbers, providing the associated computational effort involved is not excessive, and that an important aspect of mathematics, and one which pupils find attractive, is the search for pattern and regularity.

It is suggested that they be used to stimulate curiosity about numbers and their properties and so provide motivation for carrying out calculations, in order to check some prediction based on an observed regularity. Whilst some of the explorations may be "for interest only", pupils being content to observe that there is a pattern, they should be encouraged, wherever possible, to explain why the pattern is there - and to look for generalisations and further patterns. Thus the material provides opportunities for the development of skills of mathematical reasoning ("process" skills) - such as making and stating guesses (conjectures), testing, refuting or explaining conjectures, making generalisations, examining the conditions under which a statement is true, producing a (written or oral) account of activities and explanations...

In such explorations, use of the calculator may resemble experimental work in science where hypotheses are subjected to experimental test. So the calculator serves as a 'mathematics laboratory'. Thus in B2(b), for example, pupils are asked to work out

$$1^3 + 2^3,$$
$$1^3 + 2^3 + 3^3,$$
$$1^3 + 2^3 + 3^3 + 4^3,$$

which give 9, 36, 100, respectively, the squares of 3, 6, 10.
In order to predict the next result, they must either recognise the sequence of triangular numbers (see B1) or, observing the differences in

1, 3, 6, 10, ... to be 2, 3, 4, ..., conjecture that the next number in the sequence is 15, and so the next square is 225. Direct calculation then verifies this prediction, or if the first three results are not enough, provides further evidence, as much as they wish to examine on which to base a prediction. This is most concisely stated as

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = (1 + 2 + 3 + \ldots + n)^2$$

although pupils may not express their results explicitly in this form but instead perhaps as "$1^3 + 2^3 + 3^3 \ldots + n^3$ = square of n^{th} term of 1, 3, 6, 10, ..." or some statement equivalent to this. They may now <u>check</u> further values to convince themselves of its correctness. It is important both to encourage the search for pattern, so that willingness to make predictions is seen as a vital part of mathematical activity, and also to stress that a conjecture which is supported by confirming evidence, but not yet proved, remains only a promising hypothesis.

Examples B6(a), 6(b) provide instances where initial conjectures may well prove wrong, e.g. in 6(a) $n^p - n$ is a multiple of p if, and only if, p is prime, but successive conjectures may be (i) 'for all p' (disproved by p=4), (ii) 'for all <u>odd</u> p' (disproved by p=9) and (iii) 'for p prime'. Similarly, in B6(b) although $x^2 + x + 41$ is prime for all x from 0 to 39 inclusive, it is <u>not</u> prime for x=41 (which is obvious) nor for x=40 (since $x^2 + x = x(x+1)$ so that $x^2 + x$ has a factor 41 when x=40. Example B6(c) may suggest a lot of conjectures, many of which are fairly easy to explain or prove. Pupils should be encouraged to give informal 'proof-explanations', at their own level, of why they believe a result to be true, even though a rigorous formalised proof would not be appropriate for them.

Some of the computations hardly <u>require</u> the use of a calculator - but in other cases, or where pupils are insecure in computational work, a calculator can ease the burden and encourage the desire to "see if the pattern works". In a routine practice exercise, the addition 41 + 37 may be followed by another unrelated exercise such as 32 + 29, where in neither case has the pupil any vested interest in knowing what the answer is,

(apart from that of being awarded a 'mark'). In contrast, the sequence $1 + 3 = 4$, $1 + 3 + 5 = 9$, $1 + 3 + 5 + 7 = 16$ carries a strong hint that $1 + 3 + 5 + 7 + 9 = 25$ and pupils are likely to be curious to know whether this is so. See B1(b).

In any case it is hoped that use of the calculator in this way will improve both attitudes and simple computational skills without the calculator - development of the latter remains an important objective even in the age of the calculator, and it is arguable that, to use a calculator intelligently and efficiently, pupils will need to be proficient in single-digit arithmetic, and perhaps more skilled in such currently rather neglected skills as those of checking and estimating.

Index of problems

B1 Polygonal numbers

B2 Two number patterns (a) $(3x4x5x6)+1 = P.S.$ (b) $1^3 + 2^3 + 3^3 \ldots$

B3 Sums of squares $(x^2 + y^2)$

B4 Pythagorean triples

B5 Final digits

B6 3 problems on patterns, conjectures, and proof. (a) $n^p - n$, (b) $x^2 + x + 41$, (c) Table of n^2, n^3, n^4.

B7 Divisors

B8 Sums of digits (a) divisibility rules (b) a problem

B9 Sums of consecutive whole numbers

B10 Number tricks

B11, B12 } Two Number chains

Copyright © F.R. Watson, University of Keele, 1979

Note. The reproduction of Pupil material by teachers for use with their own pupils is expressly permitted.

Abbreviated Hints, Solutions and Teaching Notes for the Exercises

It is suggested that these should be consulted *after* you have tried the corresponding problems. This gives a better idea of likely pupil difficulties and may also make these brief notes more comprehensible. They are intended for the teacher, who will amplify the hints and solutions according to the needs of pupils. They are deliberately curtailed and unlikely to be intelligible directly to most pupils.

B1 (a) $T_n = \frac{n(n+1)}{2}$

As a hint, *either* ask pupils to examine the sequence of *doubles* of Δ numbers (viz. 2, 6, 12, 20, 30, 42, 56) *or* suggest they work out $\frac{T_n}{n}$

o o o o o o o (in pairs alternatively odd/even)

1 1+2 (odd +3 (even +4 (even +5

$T_n = T_{n-1} + n$, so $T_{n-1} \longrightarrow T_n$ with *switch* from odd/even or vice versa *iff* n is odd.

(b) (i) 1+3+5+7 is perfect sq., = 16. Addition of 9 as *border* gives $(4+1)^2$

Generally $n^2+(2n+1)=(n+1)^2$.

(ii) $T_n + T_{n-1}$ = square = n^2. Proof by $\frac{n(n+1)}{2} + \frac{(n-1)n}{2} = n^2$ *or* from diagram :-

$T_3 + T_4 = 4^2$

B1 continued.....

(b) (iii)

P_n	1	5	12	22	35	51	70	92
n	1	2	3	4	5	6	7	8

$P_n = 0 + \sum_{r}^{n} (3r-2) = \frac{n}{2}(3n-1)$

$\frac{P_n}{n} = 1,\ 2\frac{1}{2},\ 4,\ 5\frac{1}{2}, \ldots = \frac{3n-1}{2}$

(iv) $H_n = n(2n-1) = \frac{2n}{2}(2n-1) = T_{2n-1}$

By analogy the X-agonal numbers are given by

$X_n = 0 + \sum_{r}^{n} (\overline{x-2}r - \overline{x-3}) = \frac{n}{2}(n-1)(x-2) + n = \frac{n}{2}(\overline{x-2}n - \overline{x-4})$

Note. Some writers use an alternative definition of hexagonal numbers - see e.g. Clements, M.G., Mathematics in School 3, 3 May 1976, p. 32 and Morris, D.E., Mathematics in School 4, 2 p. 34 - so that $H_n = 3n(n-1)+1$. The successive hexagonal numbers are depicted as

1 7 19

- but this would, by analogy, make the triangular numbers 1, 4, 10, ... and the square numbers 1, 5, 13, 25

1 4 10 1 5 13

B2 (a) 5^2, 11^2, 19^2, 29^2... The answer is always a perfect square.

$n(n+1)(n+2)(n+3)+1 = (n^2+3n)(n^2+3n+2)+1 = (n^2+3n+1)^2$.

Check for n=3. (19 ✓) Some facility with "brackets" needed.

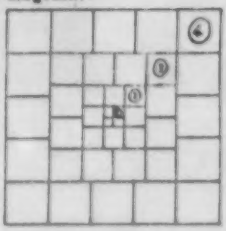
B2 (b) $9 = 3^2$, $36 = 6^2$, $100 = 10^2$ (1), 3, 6, 10 are the Δ numbers.

$$\sum_{r=1}^{n} r^3 = T_n^2 = \left[\frac{n(n+1)}{2}\right]^2 = (1+2+3+\ldots+n)^2.$$

Proof by mathematical induction, or by a diagram:-

We illustrate the case n = 4.

The border of 16 (4x4) squares surrounds a border of 12 (3x3) squares. In the central space fit 8 (2x2) squares surrounding 4 (1x1) squares.



Total area = $\left[(1+2+3+4)\times 2\right]^2$

$= 4.(4^3 + 3^3 + 2^3 + 1^3)$

(see Daxandall, P.R. & others, Proof in Mathematics, Institute of Education, University of Keele, 1978, p. 76)

B3 By trial we find:-

Expressible as $x^2 + y^2$	NO	3	7	11	19	23	31	43	47	Type 4k+3
	YES	5	13	17		29	37	41	53	Type 4k+1

It helps if pupils have done some work on modulo arithmetic. An easy systematic way of tackling the question is to make a table of values of (x^2+y^2), and see which numbers appear in the table. (i.e. to "turn the problem inside out"). We need take only x ≤ 7 since 64 is already greater than 53.

$y \backslash x$	0	1	2	3	4	. .
0	0	.....				
1	1	.....				
2	4	.....				
3	9	.....				
4	16	.....				

But pupils may best be left to try their own methods first.

The proof that every 4k+1 prime can be expressed as $x^2 + y^2$ is lengthy (see e.g. Barnard, S. and Child, J.M., Advanced Algebra, Macmillan, 1939, p. 186). However it is easy to see that no number of the form 4k + 3 (prime or not) can be expressed as $x^2 + y^2$, as follows.

Clearly x, y are not both even. If just one is odd, we have $(2k)^2 + (2\ell+1)^2 \equiv 1$ mod. 4, and if both are odd, we have $(2k+1)^2 + (2\ell+1)^2 \equiv 2$, mod. 4.

In neither case is $x^2 + y^2 \equiv 3$, mod. 4.

To take into consideration the case of composite numbers makes the situation rather more complicated - and probably too hard for this age range. (See C.3).

B4 (a) See below. There is plenty of room for experiment here, but search will be more successful if the patterns of differences, as in (b), (c), (d) are noted.

(b)

3	4	5	4 x 1
5	12	13	4 x 3
7	24	25	4 x 6
9	⋮	⋮	4 x 10
11			⋮

$2n+1,\ 2n(n+1),\ 2n^2+2n+1$

[see below; $\ell=n+1, m=n$]

(c)(d)

4	3	5	1 x 3
8	15	17	3 x 5
x = 12	35	37	5 x 7
16	⋮	⋮	⋮
20			

$4n,\ (2n-1)(2n+1),\ 2n^2+1$

[$\ell=2n, m=1$]

(4	-3	5)	
12	5	13	1 x 5
20	21	29	3 x 7
y = 28	45	53	5 x 9

$4(2n+1),\ 4n^2+4n+5$

$(4n^2+4n-3 = (2n-1)(2n+3))$

[$\ell=2n+1,\ m=2$]

3 x 5	15	8	18
3 x 7	21	20	29
3 x 9	27	⋮	⋮
	⋮	$2n(n+3)$	
	$3(2n+3)$		$2n^2+6n+9$

[$\ell=n+3,\ m=n$]

It can be shown that <u>all</u> primitive Pythagorean triples are given by $(\ell^2-m^2,\ 2\ell m,\ \ell^2+m^2)$ where ℓ, m have no common factor, and <u>one</u> of ℓ, m is even.

The differences between the largest element of a triple and one of the others, observed here are 1, 2, 8, 9, ...

Since the difference is either $(\ell^2+m^2)-(\ell^2-m^2)=2m^2$

or is $\ell^2+m^2-2\ell m=(\ell-m)^2$,

it must be a perfect square, or <u>twice</u> a perfect square. Hence,

1, 2, 8, 9, 16, 18, 25, ...

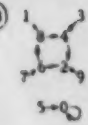
(Other differences can also arise, e.g. in Δs of form

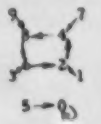
see Watson, F.R., Square triangular numbers, <u>Mathematics Teaching</u>, Dec. 1978)

B5. (x7) $0 \to 3 \to 9 \to 7 \to 1$; $4 \to 8 \to 2 \to 6 \to 4$; $5 \circlearrowright$; $0 \circlearrowright$

(x3) Like 7 but reversed. This is not because $3 \times 10 - 7$ but because $3 \times 7 \equiv 1 \bmod 10$ i.e. If $3a \equiv b(10)$ then $7b \equiv 21a \equiv a(10)$.

[7 & 3 are reciprocals, mod. 10. So are 2 & 8 for even nos. i.e. $8(2k) \equiv 2m \Rightarrow 2(2m) \equiv 2k$.]

(x8) 

(x2) 

(x9) $1 \leftrightarrow 9$; $7 \leftrightarrow 3$; $2 \leftrightarrow 8$; $4 \leftrightarrow 6$; $5 \circlearrowright$; $0 \circlearrowright$

(x4) $3 \to 2 \to 8 \to 7$; $9 \to 6 \to 4 \to 1$; $5 \to 0 \circlearrowright$

(x6) $3 \to 8 \to 4 \to 2 \to 6 \to 8$; $7 \to 2 \to 4 \to 8 \to 6 \to 2$; $1 \to 6 \to 6$; $9 \to 4 \to 6$; $5 \to 0 \circlearrowright$; $0 \circlearrowright$

(4 and 6 do not behave similarly)

See Association of Teachers of Mathematics, Mathematics in Primary Schools, C.U.P., 1967, (p. 36-44). Ours are patterns 'mod. 10' - this reference gives some patterns mod. 6, mod. 7 etc. The patterns which follow (see f, g below) are mod. 100.

(f, g) (Last two digits)

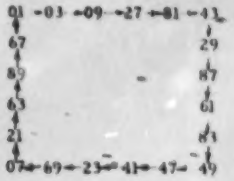
Multiplication by 7 gives cycles of length 4 e.g. $01 \to 07 \to 49 \to 43 \to 01$; $02 \to 14 \to 98 \to 86 \to 02$

There are 24 such cycles of 4 (accounting for 96 final digit pairs).

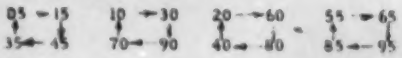
The remaining 4 are $25 \to 75 \to 25$; $50 \circlearrowright$; $00 \circlearrowright$

The explanation of this is: $7^4 = 2401 \equiv 1 \pmod{100}$. Thus $7^4 \equiv y \equiv y \pmod{100}$ and the length of every cycle is 4 or a divisor of 4 (i.e. 2 or 1).

Multiplication by 3 gives cycles of length 20. There are 4 of these, of which one is:-



There are also 4 cycles of length 4



and the remaining four digit-pairs are $25 \to 75$, $50_{\circlearrowright}$, and $00_{\circlearrowright}$.

Multiplication by 2 and 8 produces some cycles of length 20,
by 9 and 4, cycles of length 10;
by 6, cycles of length 5.
It can be proved that all cycle lengths must be divisors of $\phi(100) = 40$, where $\phi(n)$ is Euler's function. (For further details see B6a, below, and C7)

B6(a) The 'theorems' are T, T, T, F. (e.g. $n=2$, $2^9-2 = 510 \not\equiv 0 \pmod 9$.)
A first conjecture might be "n^k-n is always a multiple of k". This is true for k=2, since n(n-1) is even, but false for k=4 e.g. $2^4-2=14$.
A modified conjecture might be "n^k-n is always a multiple of k if k is odd". This clearly fails for k=9, but is true for k=11, 13, and false for k=15.
Fermat proved that "n^k-n is a multiple of k if k is prime".
There are simple direct proofs for small values of k:-
e.g. k=3, $(n-1)(n)(n+1) \equiv 0 \pmod 3$ since one of n-1, n, n+1 is a mult. of 3.
k=5, $(n-1)n(n+1)(n^2+1)$. Suppose $n \not\equiv 0, 1, 4$ i.e. $n \equiv 2$ or 3. Then
$n^2+1 \equiv 0 \pmod 5$. i.e. $n^5-n \equiv 0 \pmod 5$.
k=7, $n(n^2-1)(n^4+n^2+1)$. Suppose $n \not\equiv 0, 1, 6$ i.e. $n \equiv 2, 3, 4$ or 5,
$n^2 \equiv 4, 2, 2, 4$ and $n^4 \equiv 2, 4, 4, 2$.
Thus $n^4 + n^2 + 1 \equiv 0$ in each case.

A general proof for all p (prime) was given by Fermat:-

Let n be any residue i.e. one of 1, 2, ...(p-1).

Then 1.n, 2.n, 3.n, ...(p-1).n are all different, and so congruent in some order to 1, 2, 3, ...(p-1).

Thus $n^{p-1}.(p-1)! \equiv (p-1)!$ (mod. p), $\Rightarrow n^{p-1} \equiv 1$, mod. p i.e. $n^p \equiv n$, mod. p, including $n = p$ (†)

Euler's more general theorem is:

Let $\phi(k)$ = no. of residues, mod. k, which are prime to k.

Then if n is such a residue $n^{\phi(k)} \equiv 1$, (mod. k). (The proof is similar).

Examples k=8, $\phi(8)=4$; the 4 residues prime to 8 are 1, 3, 5, 7. Then $1^8 \equiv 3^8 \equiv 5^8 \equiv 7^8 \equiv 1$, mod. 8.

k=100, $\phi(100)=40$: Since 3 & 7 are both prime to 100, $3^{40} \equiv 7^{40} \equiv 1$, mod. 100. (In fact $7^4 \equiv 1$ and $3^{20} \equiv 1$, mod. 100 - see B5 above).

(Note. We might expect to extend Euler's result (as we did Fermat's at (†)) to give $(n^{\phi(k)}-1)n \equiv 0$, mod. k for all n. However, this is false e.g. k=4, $\phi(4) = 2$, residues 1, 3 $n^2 \equiv 1(4)$ for n = 1 or 3 but $n(n^2-1) \not\equiv 0$, mod 4 for n = 2.)

B6(b) Fails for x=40, 41; the unwary may be misled by a large number of confirming cases into believing the conjecture true.

Other polynomials which take prime values for small n are $n^2-79n+1601$ (prime for $0 \leq n < 79$; but $= 41^2$ for n = 80)

and $2x^2+29$ (prime for $0 \leq n < 28$).

Fermat suggested that $2^{2^n}+1$ is prime for all n.

The first 4 values (5, 17, 257, 65537) are all prime but

$F_5 = 2^{32}+1 = 641 \times 6700417$.

(See Courant, R. and Robbins, H., What is Mathematics, O.U.P., 1941, p. 26, and Boon, F.C., Puzzle Papers in Arithmetic, Bell, 1960, p. 60)

We can prove - "no polynomial in one variable can take only prime values" :-

Let $f(x) = \alpha_0 x^n + \alpha_1 x^{n-1} + \ldots + \alpha_n$ and let $f(b) = a$.

Then $f(b+\lambda a)$ is a multiple of a for every whole no. (and so not prime)

For $f(b+\lambda a) = \alpha_0 (b+\lambda a)^n + \alpha_1 (b+a)^{n-1} + \ldots + \alpha_n$

$= M(a) + f(b) = M(a)$.

(c.f. Baxandall, P.R. et. al., Proof in Mathematics, Institute of Education, University of Keele, 1978, p. 6)

B6(c)

n	n^2	n^3	n^4
1	1	1	1
2	4	8	16
3	9	27	81
4	16	64	256
5	25	125	625
6	36	216	1296
7	49	343	2401
8	64	512	4096
9	81	729	6561
10	100	1000	10000
11	121	1331	14641
12	144	1728	20736
13	169	2197	28561
14	196	2744	38416
15	225	3375	50625
16	256	4096	65536
17	289	4913	83521
18	324	5832	104976
19	361	6859	130321
20	400	8000	160000

Squares. Last digits 1, 4, 9, 6, 5, 6, 9, 4, 1

same in 11-19 i.e. $x^2 \ldots (10+x)^2$ have same last digit

$x^2 \equiv (10+x)^2 \pmod{10}$.

Also symmetrical i.e. $x^2 \equiv (10-x)^2 \pmod{10}$.

'gaps' in sequence are 3, 5, 7, 9, ... i.e. 1st differences (2n+1).

Cubes. Last digits repeat after 10 i.e. $x^3 \equiv (10+x)^3 \pmod{10}$.

gaps are 7, 19, 37, ... and 'gaps in gaps' - i.e. 2nd differences - are 12, 18, 24, 30....

Fourth powers. Last digits 1, 6, 1, 6, 5, 6, 1, 6, 1, 0 etc. symmetry and repetition after 10.

$x^4 \equiv (10+x)^4 \equiv (10-x)^4 \pmod{10}$. etc., etc.

B7 Number of divisors d of n

n =	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
d =	1	2	2	3	2	4	2	4	3	4	2	6	2	4	4	5	2	6	2	6	4	4	2	8	3

n =	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
d =	4	4	6	2	8	2	6	4	4	4	9	2	4	4	8	2	8	2	6	6	4	2	10	3	6

d = 2 if n prime (p); d = 3 if $n = p^2$; d = 4 if $n = p^3$ or n = pq.

$d(p^\alpha) = \alpha+1$ Proof: divisors are $1, p, p^2 \ldots p^\alpha$.

$d(p^\alpha . q^\beta) = (\alpha+1)(\beta+1)$ Proof: arrange in an array:-

1	p	p^2	p^α
q	qp	qp^2	qp^α
⋮	⋮	⋮	⋮
q^β	$q^\beta p$	.....	$q^\beta p^\alpha$

Hence $d(N) = (\alpha+1)(\beta+1)(\gamma+1)\ldots$

where $N = p^\alpha q^\beta r^\gamma \ldots$

B8(a) N is divisible by 3 (or 9) if sum of digits of N is divisible by 3 (or 9).

e.g. Let N be 'abcde' i.e. $10^4a + 10^3b + 10^2c + 10d + e =$

$9999a + 999b + 99c + 9d + (a + b + c + d + e)$

$= M(9) + (a + b + c + d + e)$

Then .H.S. $= M(3) \Leftrightarrow$ RHS $= M(3)$ i.e. $(a + b + c + d + e) = M(3)$ etc.

The rule of divisibility by 11, (a - b + c - d + e div. by 11), may be proved similarly:-

$N = 9999a + 1001b + 99c + 11d + (a - b + c - d + e)$.

B8(b) Though the results are easily found, the details of their proofs maybe too difficult for pupils.

Let N be 100a+10b+c. Then M is 99a+9b=9(11a+b), a multiple of 9.

Also $M < 999$ so its digit sum is < 27. It is therefore either 9 or 18.

For 4 and 5 digit numbers, the possible digit sums are 9, 18, 27 and 9, 18, 27, 36 respectively (proof as above).

In the 3 digit case, the digits of M will be:-

H	T	U		
a	b-1	10-b-a	provided $a+b \leq 10$ and $b \neq 0$;	sum = 9
a-1	9	10-b-a	if $a+b \leq 10$ and $b = 0$	sum = 18
a	b-2	20-b-a	if $a+b > 10$ (which $\Rightarrow b \geq 2$)	sum = 18

There are 90 pairs for a and b (a=1 to 9, b=0 to 9) of which 45 have $b \neq 0$ and $a+b \leq 10$. i.e. sum = 9 in precisely half the cases.

For a 4 digit number $1000a+100b+10c+d$, the digits of M will be

Th	H	T	U		
a	b	c-1	10-a-b-c	if $a+b+c \leq 10$, $c \neq 0$,	s = 9
a	b-1	9	10-a-b	if $a+b+c \leq 10$, $c = 0$,	s = 18
a-1	9	9	10-a	(if $b=0$, $a+b+c \leq 10$, $c = 0$,	s = 27)
a	b	c-2	20-a-b-c	if $10 \leq a+b+c \leq 20$, $c \geq 1$,	s = 18
a	b-1	9 / 8	20-a-b-c	if $10 \leq a+b+c \leq 20$, $c = 1$ / $c = 0$	s = 27
	(case b = 0 is impossible here since $a+b+c \geq 10$)				
a	b	c-3	30-a-b-c	if $20 \leq a+b+c \leq 30$, $c \geq 3$	s = 27
	(c cannot be 0, 1, 2 here since $a+b+c \geq 20$).				

B9(a) $15 = 7 + 8$; $2k+1 = k + (k+1)$.

(b) $14 \neq x + (x + 1)$ but $14 = 2 + 3 + 4 + 5$

$10 = 1+2+3+4$

$18 = 3+4+5+6$

$22 = 4+5+6+7$

(c) e.g. $18 = 5+6+7$. $3k=(k-1)+k+(k+1)$ so every multiple of 3, except 3 itself, can be so expressed.

Similarly every multiple of 5, from 15 upwards, can be expressed as the sum of 5 consecutive whole numbers

(either $(k-2)+(k-1)+k+(k+1)+(k+2)=5k$ or consider $15 = 1+2+3+4+5$

$20 = 2+3+4+5+6$ etc.)

B9 (d) But 2, 4, 8, 16 cannot be expressed as sums of consecutive integers (by trial).

Theorem. Every number N can be expressed as a sum of 2 or more consecutive integers unless $N=2^k$. No number of the form $N=2^k$ can be so expressed.

Proof.

(A) N a multiple of $2k+1 \Rightarrow$ N expressible as sum of $2k+1$ consecutive integers, provided $N > k(2k+1)$ (as above for $N=20, 2k(k+1)=5$).

(B) $N=2^p(2k+1) \Rightarrow$ N expressible as sum of 2^{p+1} consecutive integers provided $k \geq 2^p$; as above for N=14:- k-1 k k+1 k+2 $\rightarrow 2(2k+1)$ (case k=3, p=1).

Hence(C) Unless $N=2^p$, N is expressible as sum of consecutive integers. For let $N=2^p(2k+1)$; if $k \geq 2^p$, use (B), otherwise use (A).

(D) If $N=2^p$ the expression is not possible. For suppose it is, then $\sum = \frac{n}{2}(2a+(n-1).1) = \frac{n}{2}(2a+n-1)$. (Arithmetic series).

If n is odd, n-1 is even
If n is even, n-1 is odd
} in both cases N has an odd factor.

An alternative to (D) is:- The sum of 2P+1 consecutive integers is a multiple of (2P+1) - and so is odd. The sum of 2P consecutive integers is a multiple of the sum of the middle two. ..., k, k+1, ... i.e. a multiple of the odd number (2k+1).
In neither case can $N=2^p$ be the sum of consecutive integers.

B10 (a) $10(10x+1) + 7 = 100x+17$
(This example, like those of B2, B4, B7, and some which follow, illustrates the power of algebra in explaining arithmetical facts.)
In making up their own examples for their friends, pupils will practice useful manipulative skills.

B10 (b) $5(5(4x+5)+4)-145 = 100x$.

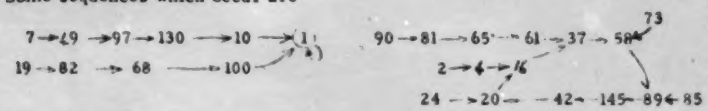
(c) $\frac{2(2x+10)-10}{2} - 2x = 5$. Result independent of x. Pupils may find similar expressions: e.g. $\frac{5(4(x+3)+4)}{20} - x = 4$

(d) $\left.\begin{matrix} 100a+10b+c \\ -100c-10b-a \end{matrix}\right\} = 100(a-c)-(a-c)=N$

which will be written as $100(a-c-1)+(90)+10-(a-c)$.

	H	T	U	
The digits of N are	a-c-1	9	10-a+c	
reversed	10-a+c	9	a-c-1	
adding	9	18	9	i.e. $900+18.10+9=1089$.

B11 Some sequences which occur are



If $n_1 < 10^{12}$, $n_2 \leq 9^2+9^2+9^2+\ldots 9^2 = 12.81 = 972$, so $n_3 < 9^2+9^2+9^2 = 243$, $n_4 \leq 9^2+9^2+1 = 163$.

Let $S = \{1, 2, 3, \ldots .163\}$ and let $x_1 \rightarrow x_2 \rightarrow x_3 \ldots$ where $x_1, x_2, x_3 \ldots \in S$

Since S is finite, $x_1, x_2, x_3 \ldots$ cannot all be different, so $x_p = x_q$, for some $p \neq q$ i.e. a cycle arises. (see the cases above; one cycle is (1)).

If we remove all the elements of the cycle from S and choose one new element y_1

$y_1 \rightarrow y_2 \rightarrow y_3 \ldots$ either joins the original cycle, or produces a new cycle.

Thus S is decomposed into cycles.

Since any starting number, however large, is soon reduced to a number $< 10^{12}$, we need only trace the behaviour of the members of S.

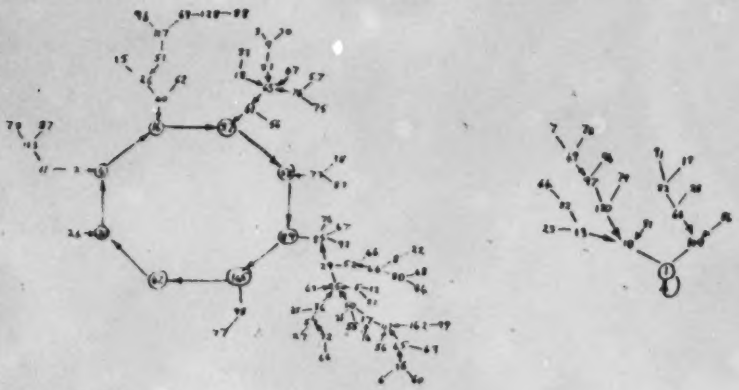
Moreover if T is the set $\{1, 2, 3, \ldots 100\}$

T is closed under this mapping, except for the elements:-

	59	69	78	79	87	88	89	95	96	97	98	99
which map to...	106	117	113	130	113	128	145	106	117	130	145	162
and then to....	37	51	11	10	11	69	42	37	51	10	42	41

Also any x in the range $101 \leq x \leq 163$ maps to a member of T,
except for $159 \to 107 \to 56$
$(149 \to 98)$

Thus it is sufficient to trace the behaviour of all $x \in T$ i.e. $x \leq 100$, as below:-



Remarks

1) If 'xy' maps to Σ, then so does 'yx'.

2) "Multiple branching" at 25 $(=5^2=3^2+4^2)$, 65$(=8^2+1^2=4^2+7^2)$, 85$(=6^2+7^2=2^2+9^2)$.

3) 83 is the 'end' of a chain, going 'outwards'. $83=4k+3 \neq x^2+y^2$
- but note $119 \to 83$, (3 digits) since $1^2+1^2+9^2=83$.

4) Indeed any number $n = \underbrace{1^2+1^2+1^2+\ldots\ldots 1^2}_{n}$

so $\underbrace{11111\ldots11}_{n} \to n$. and so no number forms the end of a chain in this sense.

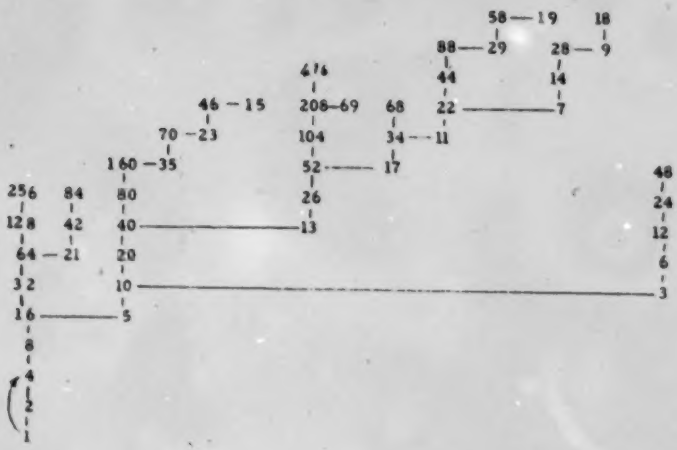
Note. This problem is also considered in Mathematics Teaching no. 85, December 1978, Jeffrey, R. "Happy numbers, sad numbers".

B12 This is an unsolved problem. (Ref. Dunn, J.A., "Arrow Chains", Mathematics Teaching, 62, March 1973).

The chain starting with 27 has 111 steps.

Other long chains arise for starting numbers 31, 41, 54, 55, 62, 63, 71, 73, 82, 91, 94, 95, 97.

Results may be displayed in the form of a tree, where vertical steps indicate 'x/2', horizontal '3x+1'. It is possible to extend the tree upwards e.g. by noting that 22 could arise, in the sequence, from 44 or from 7 - so the tree branches here.



The following BASIC program may be used to investigate these problems.

```
10   INPUT  A, B, M
20   FOR C = A TO B
30   LET I = 0
40   LET X = C
50   IF X = 2*INT(X/2) THEN 70
60   LET X = (3*X+1)*2
70   LET X = X/2
80   LET I = I+1
90   IF X>1 THEN 50
100  IF I < M THEN 120
110  PRINT C, I
120  NEXT C
130  END
```

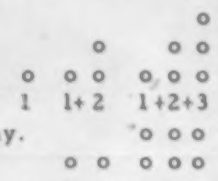
This prints out the length of all chains of length ≥ M for starting numbers in the range A ≤ C ≤ B. If we add 85 PRINT X, the members of each chain are printed. Line 60 may be altered from 3*X+1 to 3*X-1 to investigate the alternative problem - or indeed to 5*X+1 etc.....

B1 Polygonal Numbers

(a) The numbers 1, 3, 6, 10, 15, 21, ... are triangular numbers (see diagram)

Find a formula for T_n, the nth triangular number.
Hint. Try doubling the number - or look at $T_n - n$.

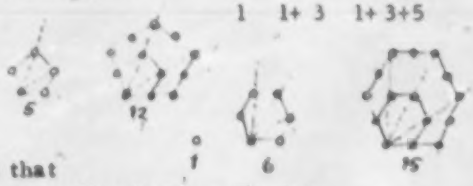
The first few numbers are odd, odd, even, even, odd, odd, . . . How does the pattern continue? Explain why.



(b) Other series are the square numbers 1, 4, 9, 16, . . .

the pentagonal numbers 1, 5, 12, . . .

the hexagonal numbers 1, 6, 15, ...



(i) From the square numbers it appears that $1 + 3 + 5 + 7 + 9$ is a perfect square. etc. Check this. Explain.

(ii) Add up any two consecutive triangular numbers. Can you explain the result? (Hint: use a Δ diagram for each of the triangular numbers)

(iii) Find all the pentagonal numbers less than 100. Find a formula for P_n, the nth pentagonal number. Check it. Hint: Look at $P_n \div n$.

(iv) Find a formula for the nth hexagonal number. Explain the connection with triangular numbers.

B3 Sums of Squares

Some numbers can be written as the sum of two squares $N = x^2 + y^2$.

17 can, $(17 = 4^2 + 1^2)$, so can 13, $(13 = 3^2 + 2^2)$.
19 cannot. (Try it).

Make a list of all the prime numbers (≥ 3) up to 53, and try to express each of them in this way. Classify them into two sets, according to whether they can or cannot be written as the sum of two squares. Can you find a rule for deciding which set a prime number should be placed in? Test your rule on a prime greater than 53.

Copyright © F.R. Watson, University of Keele, 1979

Note. The reproduction of full material by teachers for use with their own pupils is expressly permitted.

B2 Two Number Patterns

(a) Work out $(1 \times 2 \times 3 \times 4) + 1$,
$(2 \times 3 \times 4 \times 5) + 1$,
$(3 \times 4 \times 5 \times 6) + 1$.

Can you state a rule? Can you prove it?

(b) Work out $1^3 + 2^3$,
$1^3 + 2^3 + 3^3$,
$1^3 + 2^3 + 3^3 + 4^3$.

In each case the result is a square number. Can you predict the next result? Try it.
Can you state a general formula?
(Hint: B1 may help you)

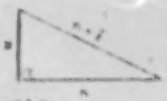
B4 Pythagorean triples

(3, 4, 5) is a Pythagorean triple (since $3^2 + 4^2 = 5^2$, so that a triangle with sides 3, 4, 5 is right-angled by the converse of Pythagoras' theorem). (6, 8, 10) is also such a triple but it is obtained by doubling all the members of the triple (3, 4, 5) so it isn't really a 'new' one.

(a) Find some more Pythagorean triples which are primitive (that is they do not have a common factor; these are the only ones we shall consider here).

(b) Consider those triples in which the largest side is one more than the next side. Arrange them in a sequence.

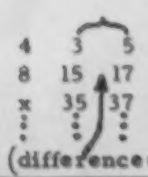
The general form is (x, n, n+1): the first few are 3 4 5
5 12 13
7 24 25

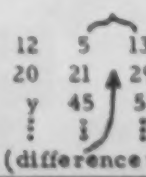


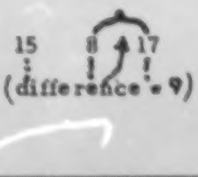
(c) Can you find any triple not in this series?
In particular, can you find other triples of the type (x, n, n+2)?
(4, 3, 5) is one.

(d) Show there are primitive triples (x, 35, 37) and (y, 45, 53)

Continue the series :-







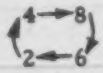
B5 Final digits

(a) Take any number ending in 9 and multiply it by 7. What is its last digit? We can write this as

$$9 \xrightarrow{\times 7} 3$$

(when we multiply by 7, a final digit of 9 changes to a final digit of 3)

(b) What happens to a number ending in 3? in 4?

What do you think this chain means? 

(c) Examine all the possible final digits and make other chains (for multiplication by 7).

(d) What patterns in final digits are there for multiplication by 3? (Do you notice anything? Can you guess why?).

(e) Now try other multiplying numbers (0, 1, 5 are not very interesting).

(f) Look for similar chain patterns in the last two digits (using 7 as a single digit multiplier).

(g) Try other single digit multipliers, looking for chains in the last two digits of the products.

B7 Divisors

(a) The number 12 is divisible by 1, 2, 3, 4, 6, 12 - so that it has 7 divisors. On the other hand, 13 is divisible by 1 and 13 only, so that it has two divisors. Make a table showing the divisors of each of the numbers from 1 to 50.

(b) Make a table showing the number of divisors of each of the numbers 1-50. Can you see any regularities in the table?
Can you devise a rule for the number of divisors of a number N?
(This is rather hard. Start with easy cases first, e.g. 31, 25, 35.
The final objective is to state a formula for the number of divisors of $N = p^a q^b r^c \ldots$ where p, q, r are the prime factors of N)

Copyright © F.R. Watson, University of Keele, 1979

Note. The reproduction of Pupil material by teachers for use with their own pupils is expressly permitted.

B6 Patterns, conjectures and proofs - 3 problems

(a) Investigate the "theorems": (i) $n^3 - n$ is always a multiple of 3
(ii) $n^5 - n$ is always a multiple of 5
(iii) $n^7 - n$ is always a multiple of 7
(iv) $n^9 - n$ is always a multiple of 9

Generalise.

(b) Evaluate the expression $x^2 + x + 41$ for $x = 0, 1, 2, 3....$
Is the result always prime?

(c) Make a table of squares (cubes, fourth powers) of the numbers from 1 - 20. What patterns occur in the table and how may they be explained?

B8 Sums of digits

(a) (i) Which of the following numbers are divisible by 3?

879, 3762, 28795, 7826, 2311, 6134, 3213, 9859, 7290

(ii) Which of the numbers in (i) are divisible by 9?

(iii) Make a table as follows

Number	Sum of digits	Divisible by 3?	by 9?
879	8 + 7 + 9 = 24	yes	no
3762	18	yes	yes
⋮	⋮	⋮	⋮

Can you see any connection between divisibility by 3 or 9 and the sum of the digits?

(b) A problem: Start by choosing any whole number over 100 but less than 1000 (that is, a number of three digits) Call it N. Add up the three digits of your number N, and subtract the answer from your original number. This gives you a new number - call it M. Add up the digits of your new number M. The answer is often 9. Try choosing some other starting numbers for N to see if you can get a different answer at the end. See what other answers you can find. Now try this with numbers which have four digits, 5 digits, etc.

B9 Sums of consecutive whole numbers

(a) 15 can be written as the sum of two consecutive whole numbers $15 = 7 + 8$. Show how this can be done with any odd number.

(b) Can you write 14 as the sum of two consecutive whole numbers? Of 4 consecutive whole numbers? Try other even numbers 10, 18, 22, ...

(c) "Any multiple of 3 can be written as the sum of 3 consecutive whole numbers". Is this true? (Try it. Prove your statement if you can)

(d) $15 = 1 + 2 + 3 + 4 + 5 = 4 + 5 + 6 = 7 + 8$. 15 can be written in several ways as the sum of consecutive whole numbers. 6 can be written in only <u>one</u> way $6 = 1 + 2 + 3$. Can <u>every</u> number be written in <u>some</u> way as the sum of consecutive whole numbers?

A Number Chain

B11

Take a number n_1 and form a new number n_2 equal to the sum of the squares of its digits.
Now use n_2 to form a new number n_3 in the same way.

Here is an example: $n_1 = 43$. $n_2 = 4^2 + 3^2 = 25$.

$43 \rightarrow 25 \rightarrow 29 \rightarrow 85 \rightarrow 89 \rightarrow \ldots\ldots\ldots$

Continue this example. You will find that you come back to 89 again.
Try to find out whether there are other loops like this.

Copyright © P.R. Watson, University of Keele, 1979

Note. The reproduction of Pupil material by teachers for use with their own pupils is expressly permitted.

B10 Number tricks

(a) Choose any number between 1 and 9 inclusive. Multiply it by 10 and add 1. Then multiply your answer by 10 and add 7.
Try this again, starting with a different number. What do you notice?
Try some more numbers to see if your idea works every time.
(You can pretend to be a mind reader!)
Can you see why this trick always works?
Make up another trick like this one.

(b) Think of a number between 1 and 10. Multiply it by 4, add 5, multiply by 5 and add 4, and finally multiply by 5. Subtract 145.
Try this with other starting numbers. What do you notice?
Can you explain it? Make up another similar trick.

(c) Think of a single digit number, (that is one between 1 and 10), double it, add 10 and double the answer. Subtract 10 and halve the result.
Now take away twice your original number. What is your answer?
Try this with another starting number. Do you notice anything about your answer? Try to explain how this trick works. Make up another one like it.

(d) Another well-known trick is this -
Take any three digit number, in which the units digit and the hundreds digit differ by at least 2. Reverse the digits of your number, and subtract the smaller of these two numbers from the larger. This gives you a new number, N. Reverse the digits of N, and add the answer to N. You will get 1089 whatever number you start with. Try it!
** To explain why this happens is not too easy - see if you can.

B12 Another Number Chain

Start with any number. If it is even, then halve it. If it is odd, treble it and add 1. Then follow the same instructions. That is

if U_n is even then $U_{n+1} = \frac{1}{2}U_n$; if U_n is odd then $U_{n+1} = 3U_n + 1$.

Example: $3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

This sequence was obtained by following these instructions, (starting with the number 3).

This sequence always seems to end up at 1 eventually (No one has yet been able to prove that this must happen!) Try it.

(Another rule which gives similar 'chains' is like the one above, except that if U_n is odd then $U_{n+1} = 3U_n - 1$).

Exploring Numbers - Number Patterns and Investigations

Pupil's Activities. (Series B)

(Some suggestions for using the electronic calculator)

(A set of Teachers' notes and solutions is available)

Copyright © F.R. Watson, University of Keele, 1979

Note. The reproduction of Pupil material by teachers for use with their own pupils is expressly permitted.

EXPLORING NUMBERS

Some Investigations with a Calculator

Series C

TEACHERS' NOTES

F.R. Watson
Institute of Education
University of Keele

Copyright © F.R. Watson, University of Keele, 1979

Note The reproduction of Pupil material by teachers for use with their own pupils is expressly permitted.

The advent of the 'pocket' calculator seems likely to have a profound effect on mathematics teaching before long. One way in which it can be used immediately, without any change in 'syllabus', is as a means of exploration and investigation of number properties and patterns.

It is hoped these suggestions will be a useful starting point for teachers, who will modify and expand them to suit their own circumstances. I should welcome any comments, and notes of errors which, perhaps inevitably, will have escaped my attention.

Several people have helped in the production of these materials, but I should particularly like to thank Mrs. B. Wiggins and Mrs. C. Lea for their contributions in typing and printing respectively.

F.R. Watson
February 1979

Teaching Notes, with hints and solutions, for Number patterns and investigations. Series C. (16 - 19 years)

Introduction

In these problems the main interest will be in proving properties suggested by work with the calculator. In many instances the calculator is not strictly necessary, but partly serves to reduce the labour of calculation and checking. Further, in the alternation of conjecture, checking, verification and proof, the 'experiment' and the 'explanation' go hand in hand, experiment suggesting conjectures which require explanation and give rise to predictions, which in turn lead to further experiment.

The following quotation, from a recent article, amplifies this point:
"(The investigation) illuminates the interplay between the power of the calculator in performing routine operations and the power of the human mind in deciding which operations to perform. The calculator supercedes much of arithmetic, but it does not replace mathematics; indeed its use can lead us into mathematical 'discoveries' which, without it, we would be unlikely to hit upon.... We have used the calculator to explore properties, suggest conjectures, check predications. Without it, some of this work would have been tedious, and perhaps the exploration and checking would have been curtailed. We have also demonstrated how mathematical methods can immensely enhance the power of the calculator, by illustrating the interplay between 'brain-power' and 'calculator power'. Such instances suggest that the impact of the calculator on mathematics teaching will not be restricted to making statistical calculations easier or to rendering logarithms obsolete as a calculating device, but that it is likely to bring about more far-reaching changes in the way we look at mathematics, and in the way in which certain topics are approached."

(From "Square Triangular Numbers", a calculator investigation - which takes up and extends the problem posed in 1(b)(vi) below - though at a level rather more difficult than the material here. See M. Teaching 85, 86, and 87)

Copyright © F.R. Watson, University of Keele, 1979

Note. The reproduction of Pupil material by teachers for use with their own pupils is expressly permitted.

The Material

A list of topics covered in the pupil material is given below. The hints and notes which follow may be useful to teachers as a supplement to their own solutions of the problems - as background or in showing teaching points which can arise from the material.

Contents List of the pupil material

1. Triangular, square numbers, etc.
2a. Products of consecutive integers.
2b. Σr^3 and Σr^2.
3. Sums of two squares.
4. Pythagorean triples.
5. Fibonacci sequence.
6. Recurring decimals.
7. Final digits of products.
8. The divisors of N.
9. Squares, cubes.
10. Divisibility rules.
11. A property of $2^{p-1}-1$.
12. Sums of consecutive integers.
13. Multiple-precision working.
14. A property of the sequence $x_{n+1} = x_n(2 - ax_n)$.

Other problems suitable for pupils at this level may arise from seeking proofs - and generalisations with further proofs - of some of the properties investigated in the pupil material, Series A and B.

e.g. "If a two digit number is multiplied by 9 the digit sum is 9 or 18". Prove this and determine a rule for the digit sum. Extend this to 3 digit numbers. (*See 4.2*).

Other Series A problems which might be examined similarly include *15d*), *16f*, 17, 19a.

The Series B material contains simplified versions of some of the problems in this set. However, other problems not included here, which might provide material for extension and generalisation are:-

6(a) In what circumstances is $n^k - n$ a multiple of k.

6(b) Is $x^2 + x + 41$ always prime?

8(b) A digit-sum problem.

10(a, b, c, d). Number tricks - explain why these work.

11, 12. Arrow chains. $100a + 10b + c \rightarrow a^2 + b^2 + c^2 \rightarrow$

and $u_{n+1} = \frac{1}{2}u_n$, u_n even

or $= 3u_n + 1$, u_n odd.

Abbreviated Hints, Solutions and Teaching Notes for the Exercises

It is suggested that these should be consulted after you have tried the corresponding problems. This gives a better idea of likely pupil difficulties and may also make these brief notes more comprehensible. They are intended for the teacher, who will amplify the hints and solutions according to the needs of pupils. They are deliberately curtailed and unlikely to be intelligible directly to most pupils.

C1 (a) $T_n = \frac{n(n+1)}{2}$

As a hint, either ask pupils to examine the sequence of doubles of A numbers (viz. 2, 6, 12, 20, 30, 42, 56) or suggest they work out $T_n + n$

o o o o o o o (in pairs alternatively odd/even)

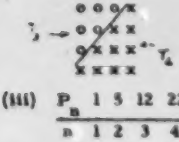
1 1+2 | odd +3 | even +4 | even +5

$T_n = T_{n-1} + n$, so $T_{n-1} \longrightarrow T_n$ with switch from odd/even or vice versa iff n is odd.

(b) (i) 1+3+5+7 is perfect sq., = 16. Addition of 9 as border gives $(4+1)^2$

Generally $n^2+(2n+1)=(n+1)^2$.

(ii) $T_n + T_{n-1}$ = square = n^2. Proof by $\frac{n(n+1)}{2} + \frac{(n-1)n}{2} = n^2$ or from diagram



$T_3 + T_4 = 4^2$

(iii)

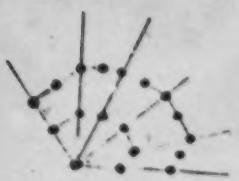
P_n	1	5	12	22	35	51	70	92
n	1	2	3	4	5	6	7	8

$P_n = 0 + \sum_{r}^{n}(3r-2) = \frac{n}{2}(3n-1)$

$\frac{P_n}{n} = 1,\ 2\frac{1}{2},\ 4,\ 5\frac{1}{2}, \ldots = \frac{3n-1}{2}$.

(iv) $H_n = n(2n-1) = \frac{2n}{2}(2n-1) = T_{2n-1}$

(v)



Illustrated is the 3rd 7-agonal number.

This is 1 + 6 + 11 (examining successive 'borders').

In general, the nth X-agonal is

$1 + \overbrace{1.(X-2)+1}^{} + \overbrace{2(X-2)+1}^{} + \overbrace{3(X-2)+1}^{} \ldots$

$+ \overbrace{(n-1)(X-2)+1}^{} = 1 + (n-1) + (X-2)\sum_{r=1}^{n-1} r$

$= n + \frac{n-1}{2} n(X-2)$

(using the successive borders)

(vi) $35^2 = 1225 = 5^2 . 7^2 = \frac{49.50}{2}$

$204^2 = 12^2 . 17^2 = 41616 = \frac{288.289}{2}$

} etc.

See Watson, F.R., 'Square Triangular Numbers'. Mathematics Teaching, 85, Dec. 1978, for a further discussion.

Note. Some writers use an alternative definition of hexagonal numbers - see e.g. Clements, M.G., Mathematics in School 3, 3 May 1976, p. 32 and Morris, D.E., Mathematics in School 4, 2 p. 31 - so that $H_n = 3n(n-1)+1$. The successive hexagonal numbers are depicted as

0 1 7 19

- but this would, by analogy, make the triangular numbers 1, 4, 10, ... and the square numbers 1, 5, 13, 25

1 4 10 1 5 13

C2. (a) (i) $n(n+1)$ is even, since one of n, $n+1 \equiv 0 \bmod 2$.

(ii) $n(n+1)(n+2)$ is even and a mult. of 3, since one of n, $n+1$, $n+2 \equiv 0 \bmod 3$.

(iii) $n(n+1)(n+2)(n+3)$ is a mult. of 3 and of 8, since 2 numbers are even, and one is a multiple of 4.

(iv) $n(n+1)(n+2)(n+3) = (n^2 + 3n + 1)^2 - 1$.

2 (b)(i) $9 = 3^2$, $36 = 6^2$, $100 = 10^2$ (1), 3, 6, 10 are the Δ numbers.

$$\sum_{1}^{n} r^3 = T_n^2 = \left[\frac{n(n+1)}{2}\right]^2 = (1+2+3+\ldots+n)^2.$$

Proof by mathematical induction, or by a diagram:-

We illustrate the case n = 4.

The border of 1 + 3 + 5 + 7 (= 4²) squares surrounds a border of 12 (3x3) squares. In the central space fit 8 (2x2) squares surrounding 4 (1x1) squares.

Total area = $\left[(1 + 2 + 3 + 4) \times 2\right]^2$

= $4.(4^3 + 3^3 + 2^3 + 1^3)$

(see Baxandall, P.R. & others, Proof in Mathematics, Institute of Education, University of Keele, 1978, p.76)

(ii) 6 x sum = 6 30 84 180 330 546

1x2x3 2x3x5 3x6x7 4x5x9 5x6x11 6x7x13 n(n+1)(2n+1)

Conjecture $S = \frac{1}{6}\, n(n+1)(2n+1)$. Proof by mathematical induction - or see textbooks.

Alternatively, factorising the sum we note factors :-

1, 5 , 2 x 7 , 3 x 6 , 5 x 11 , 7 x 13 , 9x20 , 12x17

& so examine $\frac{6}{2n+1}$ = 1, 1, 2, $\frac{10}{3}$, 5, 7, $\frac{28}{3}$, 12

or $\frac{3S}{2n+1}$ = 1, 3, 6, 10, 15, 21, 28, 36 (Δ nos.)

So $S = \frac{1}{3}(2n+1).T_n = \frac{1}{3}.\frac{n(n+1)}{2}.(2n+1)$

C3 By trial we find:-

(a,b) Expressible as $x^2 + y^2$

NO	3	7	11	19	23		31		43	47		Type 4k+3
YES	5		13	17		29		37	41		53	Type 4k+1

It helps if pupils have done some work on modulo arithmetic. An easy systematic way of tackling the question is to make a table of values of (x^2+y^2), and see which numbers appear in the table. (i.e. to "turn the problem inside out"). We need take only $x \leq 7$ since 64 is already greater than 53.

y \ x	0	1	2	3	4	...
0	0	.....				
1	1	.....				
2	4	.....				
3	9	.....				
4	16	.....				

But pupils may best be left to try their own methods first.

The proof that every 4k+1 prime can be expressed as $x^2 + y^2$ is lengthy (see e.g. Barnard, S. and Child, J.M., Advanced Algebra, Macmillan, 1939, p.186). However it is easy to see that no number of the form 4k + 3 (prime or not) can be expressed as $x^2 + y^2$, as follows.

Clearly x, y are not both even. If just one is odd, we have $(2k)^2 + (2\ell+1)^2 \equiv 1$ mod. 4, and if both are odd, we have $(2k+1)^2 + (2\ell+1)^2 \equiv 2$, mod. 4.
In neither case is $x^2 + y^2 \equiv 3$, mod. 4.

(c) It can be shown that if N is composite, $N = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} q_1^{\beta_1} q_2^{\beta_2} \ldots q_s^{\beta_s}$ then $N = x^2 + y^2$ iff β_i is even for all i. Thus $9 = 3^2 + 0^2$,
$45 = 3^2.5 = 6^2 + 3^2$, $125 = 9^2.5 = 121 + 4 = 11^2 + 2^2$.
i.e. If N contains factors of type q these must appear as squares.
In this context, $2 = 1^2 + 1^2$ behaves as a prime of type 'p'.
It may be helpful to consider in order i) $N = q_1^2 . q_2^2 \ldots\ldots$
ii) N odd, iii) $N = 2^k p_1 p_2 p_3 \ldots$, iv) General N.

C4 (a) See below. There is plenty of room for experiment here, but search will be more successful if the patterns of differences, as in (b), (c), are noted.

(b)

3	4	5	4 x 1
5	12	13	4 x 3
7	24	25	4 x 6
9	⋮	⋮	4 x 10
11		⋮	

$2n+1,\ 2n(n+1),\ 2n^2+2n+1$

[see below; $\ell = n+1,\ m = n$]

(c)

4	3	5	1 x 3
8	15	17	3 x 5
x = 12	35	37	5 x 7
16	⋮	⋮	⋮
20			

$4n,\ (2n-1)(2n+1),\ 2n^2+1$

[$\ell = 2n,\ m = 1$]

C4

(c) (cont.)

(4	-3	5)	
12	5	13	1 x 5
20	21	29	3 x 7
y= 28	45	53	5 x 9
	$4(2n+1)$	$4n^2+4n+5$	

$(4n^2+4n-3=(2n+1)(2n+3))$

$[\ell=2n+1,\ m=2]$

3 x 5	15	8	18
3 x 7	21	20	29
3 x 9	27		

$2n(n+1)$

$3(2n+3)$ $2n^2+6n+7$

$[\ell=n+3,\ m=n]$

It can be shown that all primitive Pythagorean triples are given by $(\ell^2-m^2,\ 2\ell m,\ \ell^2+m^2)$ where ℓ, m have no common factor, and one of ℓ, m is even.

The differences between the largest element of a triple and one of the others observed here are 1, 2, 8, 9, ...

Since the difference is either $(\ell^2+m^2)-(\ell^2-m^2)=2m^2$

or is $\ell^2+m^2-2\ell m=(\ell-m)^2$,

it must be a perfect square, or twice a perfect square. Hence,

1, 2, 8, 9, 16, 18, 25, ...

(Other differences can also arise, e.g. in Δs of form

see Watson, F.R., Square triangular numbers, Mathematics Teaching, Dec. 1978)

(d) If two members of a triple have a common factor h, then h is a factor of the third member and so the triple is not primitive.

If two are even, so is the third; if two are odd, the third must be even.

Hence just two odd, one even.

$(\text{Odd})^2+(\text{Odd})^2 \neq (\text{even})^2$ (congruence mod. 4; $1+1 \not\equiv 0$ mod. 4).

i.e. largest must be odd. Hence numbers are x = odd, y = 2n, z = odd.

(e) N = 60, by trial.

(f) Let $x=\ell^2-m^2$, $y=2\ell m$, $z=\ell^2+m^2$, and m be even.

Then, $2\ell m \equiv 0(4)$, and if neither of ℓ, m $\equiv 0$ mod. 3 then either $\ell-m$ or $\ell+m$ is. Hence $x \equiv 0(3)$.

If none of ℓ, m, $\ell-m$, $\ell+m \equiv 0$ mod. 5, they must be 1| 1| 4| 4|
2| 3| 2| 3|

and in each case $\ell^2+m^2 \equiv 0(5)$.

71

Hence, xyz divisible by 4x3x5 = 60. (Since this is attained for (3, 4, 5), no larger number exists with this property)

(cc) Proof of result given in (f).

$y^2 = (a-x)(a+x) = 4k^2$ since y is even.

Now if $\left.\begin{matrix} a-x \\ a+x \end{matrix}\right\}$ have common factor 2, this is a factor of 2a and of 2x

i.e. (x, y, a) not primitive

Therefore $\left.\begin{matrix} a+x = 2l^2 \\ a-x = 2m^2 \end{matrix}\right\} \rightarrow \begin{matrix} a = l^2 + m^2 \\ x = l^2 - m^2 \end{matrix}$ and $y = 2lm$.

l, m have no common factor, thus at most one is even.

If both odd, x is even and so triple non-primitive.

Therefore, just one of l, m is even.

CS. (a) To prove $u_1 + u_2 + \ldots + u_n = u_{n+2} - 1$. This is true for n = 1. If true for n = k then $(u_1 + u_2 + \ldots u_k) + u_{k+1} = u_{k+2} - 1 + u_{k+1} = u_{k+3} - 1$.

Hence etc. by M. Induction.

Alternatively, "borrow" $1 :- 1 + u_1 = u_3$

$(1 + u_1) + u_2 + u_3 + u_4 \ldots + u_n$

$= (u_3 + u_2) + u_3 + u_4 + \ldots$

$= (u_4 + u_3) + u_4 + \ldots$

$= (u_5 + u_4) + \ldots + u_n$

$= \ldots\ldots\ldots\ldots$

$= u_{n+1} + u_n = u_{n+2}$.

(b) To prove $u_1 + u_3 + u_5 + u_7 \ldots u_{2n+1} = u_{2n+2}$. For LHS $= u_2 + u_3 + u_5 \ldots$

$= u_4 + u_5 \ldots = u_6 + u_7 \ldots$ etc. (Or by M. Induction).

(c) To prove $u_1^2 + u_2^2 + \ldots + u_n^2 = u_n u_{n+1}$. True for n = 1.

If $\sum_{r}^{k} u_r^2 = u_k u_{k+1}$, $\sum_{r}^{k+1} = u_{k+1}(u_{k+1} + u_k) = u_{k+1} u_{k+2}$

(d) True, by M.I. When k = 1 this is $u_{n+1} = u_n + u_{n-1}$, and for k = 2,

$u_{n+2} = u_{n-1} + u_n \cdot 2$, which is true.

When k = h, suppose both $\left\{\begin{matrix} u_{n+h} = (u_{n-1} u_h) + (u_n \cdot u_{h+1}) \\ u_{n+h-1} = (u_{n-1} u_{h-1}) + (u_n u_h) \end{matrix}\right.$ are true.

and

72 Adding gives case k = (h+1). Hence by M.I.

Here we assume the result true for two values of k, namely h and h-1, and prove it true for k = h + 1.

c.f.(cont.) (e) To prove $u_2 + u_4 + \dots + u_{2n} = u_{2n+1} - 1$. This follows by subtracting

$$\left.\begin{aligned} u_1 + u_2 + u_3 + \dots + u_{2n+1} &= u_{2n+3} - 1 \\ \& \; u_1 + u_3 + u_5 \dots + u_{2n+1} &= u_{2n+2} \end{aligned}\right\} \begin{matrix}(a)\\(b)\end{matrix}$$ above.

(f) To prove $u_{n+1}^2 - u_{n-1}^2 = u_{2n}$. LHS $= (u_{n+1} - u_{n-1})(u_{n+1} + u_{n-1}) = u_n(u_{n+1} + u_{n-1}) = (u_n u_{n+1}) + (u_n \cdot u_{n-1}) = u_{2n}$ using (d), with n for h.

(g) To prove $\frac{1}{2}(u_{n+3} - u_n) = u_{n+1}$ which is equivalent to $u_{n+2} + u_{n+1} - u_n = 2u_{n+1}$ which is true.

(h) $u_3 = 2$. u_6, u_9, u_{12} all multiples of 2.

Similarly $u_4 = 3$, u_8, u_{12}, ... all multiples of 3.

Conjecture: u_{sk} is a multiple of u_s

Proof by induction on k, using (d). True for $k = 1$. Suppose true when $k = h$.

$$u_{s(h+1)} = u_{sh+s} = \underbrace{\left(u_{sh-1} u_s\right)}_{\text{mult. of } u_s} + \left(u_{sh} u_{s+1}\right)$$

(i) If u_n and u_{n+1} have common factor h, this is also a c.f. of u_{n-1}, u_n - and so of u_1, u_2 - i.e. is 1.

Similarly u_n, u_{n+2} have common factor $h \neq h$ is factor of u_{n+1} (and $\therefore$ h is 1).

(j) Conjecture h.c.f. $(u_m, u_n) = u_{hcf(m,n)}$

Proof. Let hcf(m, n) be α. Then u_α is a factor of u_m and of u_n (by part (h) above)

Let $\left.\begin{aligned} m &= p\alpha \\ n &= q\alpha \end{aligned}\right\} m - n = s\alpha$. Then using (d), $u_m = \left(u_{n-1} u_{m-n}\right) + u_n u_{m-n+1}$

i.e. $u_{\alpha p} = \left(u_{\alpha q - 1} \cdot u_{\alpha r}\right) + \left(u_{\alpha q} \cdot u_{\alpha r + 1}\right)$

so any factor of $u_{\alpha q}$ and $u_{\alpha r}$ is factor of $u_{\alpha s}$ and so down to u_α.

Thus u_α is highest c.f.

(k) u_m divides u_n $\Rightarrow$ h.c.f. $(u_m, u_n) = u_m$. i.e. m = h.c.f. (m, n) i.e. $m \mid n$. ("m is divisor of n")

C5. (i) Mod. 7 remainders are

1 1 2 3 5 1 6 0 6 6 5 4 2 6 1 0 | 1 1 2 3 5 ...

period length = 16.

Remainder pattern, (mod. n) must repeat with period $\leq n^2 - 1$,
since once a pair of remainders repeat, the whole sequence repeats.
There are at most $n^2 - 1$ distinct pairs, (since 0, 0) cannot be a pair).
Actual periods are:-

n	2	3	5	7	11	13	17	19	23	29	31
period	3	8	20	16	10	28	36	18	48	14	30

Except for the value n = 5, period is related to (n-1) $\{3, 11, 19, 29, 31 \ldots\}$
or to (n+1) $\{2, 7, 13, 17, 23 \ldots\}$

(For a proof of this result see Dynkin E. B. & Uspenskii, V. A., Problems in the theory of numbers, D. C. Heath, 1963.)

C6. Results are stated here. For proofs see Watson, F. R. "Finding out about recurring decimals", Mathematics in School, 5, 2., March 1976.

(b) $\frac{1}{13} = .076923..$ $\frac{2}{13} = .153846..$

$\frac{3}{13} = .230769.$

$\{1, 3, 4, 9, 10, 12\}$ $\{2, 5, 6, 7, 8, 11\}$

(c) $\frac{1}{39} = 0.0256410$. 4 sets of 6:

$\{1, 4, 10, 16, 22, 25\}$ $\{2, 5, 8, 11, 20, 32\}$ $\{7, 31, 34, 37, 19, 28\}$ $\{14, 23, 38, 35, 29, 17\}$

$\frac{1}{13}$ of $\frac{1}{3}$: $\{1, 12, 30, 9, 27, 36\}$ $\{6, 24, 21, 18, 15, 33\}$ $\frac{1}{3}$ of $\frac{1}{13}$: $\{13, 26\}$

(c) $\frac{1}{41} = 0.02439..$ 8 sets of 5. $\{1, 10, 18, 16, 37\}$ $\{2, 20, 36, 32, 33\}$
$\{3, 30, 13, 7, 29\}$ $\{4, 40, 31, 23, 25\}$
$\{5, 9, 8, 39, 21\}$ $\{6, 19, 26, 14, 17\}$
$\{11, 28, 34, 12, 28\}$ $\{15, 27, 24, 35, 22\}$

(d) Terminating if N has only 2 and 5 as factors.
Pure recurring if N has no factors 2, 5. } Mixed otherwise.

(e) λ is a divisor of (p-1).

C6. (f)

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\phi(n)$	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8	16	6	18	8

$\phi(p) = p-1$

$\phi(p^2) = p(p-1)$

$\phi(pq) = (p-1)(q-1)$

$\phi(ab) = \phi(a).\phi(b)$ where a, b are prime.

(g) λ is a divisor of $\phi(n)$.

C7

(x7) $9 \to 3 \to 7 \to 1 \to 9$; $4 \to 8 \to 2 \to 6 \to 4$; $5 \circlearrowright$; $0 \circlearrowright$

(x3) Like 7 but reversed. This is not because $3 \times 10 - 7$ but because $3 \times 7 \equiv 1 \bmod 10$ i.e. If $3a \equiv b(10)$ then $7b \equiv 21a \equiv a(10)$.

[7 & 3 are reciprocals, mod. 10. So are 2 & 8 for even nos. i.e. $\theta(2k) \equiv 2m \Rightarrow 2(2m) \equiv 2k$.]

(x8) $1 \to 3 \to 9 \to 7 \to 1$; $2 \to 6 \to 8 \to 4 \to 2$; $5 \to 0 \circlearrowright$

(x2) $9 \to 7 \to 3 \to 1 \to 9$; $4 \to 8 \to 6 \to 2 \to 4$; $5 \to 0 \circlearrowright$

(x9) $1 \leftrightarrow 9$; $7 \leftrightarrow 3$; $2 \leftrightarrow 8$; $4 \leftrightarrow 6$; $5 \circlearrowright$; $0 \circlearrowright$

(x4) $3 \to 2 \to 8 \to 7 \to 3$; $9 \to 6 \to 4 \to 1 \to 9$; $5 \to 0 \circlearrowright$

(x6) $3 \to 8 \to 3$; $7 \to 2 \to 7$; $1 \to 6 \to 1$; $9 \to 4 \to 9$; $5 \to 5$; $0 \to 0$

(4 and 6 do not behave similarly)

See Association of Teachers of Mathematics, Mathematics in Primary Schools, C.U.P. 1967, (p. 36-44). Ours are patterns 'mod. 10^n' - this reference gives some patterns mod. 6, mod. 7 etc. The patterns which follow (see f, g below) are mod. 100.

7 (f,s) Multiplication by 7 gives cycles of length 4 e.g. $01 \to 07 \to 49 \to 43 \to 01$ and $02 \to 14 \to 98 \to 86 \to 02$

There are 24 such cycles of 4 (accounting for 96 final digit pairs).

The remaining 4 are $25 \leftrightarrow 75$ 50 00

The explanation of this is: $7^4 = 2401 \equiv 1 \pmod{100}$. Thus $7^4 \times y \equiv y \pmod{100}$ and the length of every cycle is 4 or a divisor of 4 (i.e. 2 or 1).

Multiplication by 3 gives cycles of length 20. There are 4 of these, of which one is:-

$$01 \to 03 \to 09 \to 27 \to 81 \to 43 \to 29 \to 87 \to 61 \to 83 \to 49 \to 47 \to 41 \to 23 \to 69 \to 07 \to 21 \to 63 \to 89 \to 67 \to 01$$

There are also 4 cycles of length 4

$$05 \to 15 \to 35 \to 45 \to 05 \qquad 10 \to 30 \to 70 \to 90 \to 10 \qquad 20 \to 60 \to 40 \to 80 \to 20 \qquad 55 \to 65 \to 85 \to 95 \to 55$$

and the remaining four digit-pairs are $25 \leftrightarrow 75$, 50, and 00.

Multiplication by 2 and 8 produces some cycles of length 20;

by 9 and 4, cycles of length 10;

by 6, cycles of length 5.

It can be proved that all cycle lengths must be divisors of $\phi(100) = 40$, where $\phi(n)$ is Euler's function. (see C 6.).

(h) Modulo 10, we have $7 \to 7^2 \equiv 9, \to 7^3 \equiv 3 \to 7^4 \equiv 1$ (since $7^4 = 2401$).

Hence cycle is of length 4; the cycle containing 2, 4, 8, 6 is the 'double' of this.

Similarly, $3^4 = 81 \equiv 1 \pmod{10}$, and so multiplication by 3 produces similar cycles of 4.

[These are reversed since $3 \times 7 \equiv 1 \pmod{10}$ i.e. 3 and 7 are reciprocals].

Multiplication by 8 or 2 gives cycles of 4 ($2^5 \equiv 2$, $8^5 \equiv 8 \pmod{10}$ - though $2^4 \not\equiv 1 \not\equiv 8^4$, so odd nos. do not form a cycle).

Multiplication by 4 gives cycles of 2 since $4^3 \equiv 4$; $6^2 \equiv 6$, so that 6 has 'cycles of 1'.

Since 2 is a factor of 10, multiplication by an even number is "irreversible"; thus the property of 'being even' is preserved mod. 10, whereas the property of "being a multiple of 3" is not preserved :-

e.g. $3 \to 9 \to 7$ (×3 ×3)

C7 (i) Since $7^4 = 2401 \equiv 1 \bmod 100$ the chains for x7 are of length 4.
Any starting no. $x \xrightarrow{7} 7x$ gives a chain of length 4 (or 2 or 1 - sub-multiples of 4).
But $3^4 = 81 \not\equiv 1 \bmod 100$. $81^2 \equiv 61$, $81^3 \equiv 41$, $81^4 \equiv 21$, $81^5 \equiv 01$ so that $3^{20} \equiv 1 \bmod 100$ and 'x3' gives cycles of 20. (or sub multiples - possibly 10, 5, 4, 2, 1).
Some results for single digit multiples, mod. 100, are given below: (x 0)

C8(a, b) Number of divisors d of n

n =	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
d =	1	2	2	3	2	4	2	4	3	4	2	6	2	4	4	5	2	6	2	6	4	4	2	8	3

n =	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
d =	4	4	6	2	8	2	6	4	4	4	9	2	4	4	8	2	8	2	6	6	4	2	10	3	6

d = 2 if n prime (p); d = 3 if $n = p^2$; d = 4 if $n = p^3$ or $n = pq$.
$d(p^\alpha) = \alpha+1$ Proof: divisors are $1, p, p^2 \ldots p^\alpha$.
$d(p^\alpha . q^\beta) = (\alpha+1)(\beta+1)$ Proof: arrange in an array:-

1	p	p^2	p^α
q	qp	qp^2	qp^α
⋮	⋮	⋮	⋮
q^β	$q^\beta p$	.....	$q^\beta p^\alpha$

Hence $d(N) = (\alpha+1)(\beta+1)(\gamma+1)\ldots$
where $N = p^\alpha q^\beta r^\gamma \ldots$

C8. (c) Divisors of pq are 1, p, q, pq with sum (1+p)(1+q):
of p^r are $1, p, p^2, p^3 \ldots p^r$, with sum $\frac{(1-p^{r+1})}{1-p}$

Sum of divisors of $(N = p^\alpha q^\beta r^\gamma \ldots)$ is $\frac{(p^{\alpha+1}-1)}{p-1} \cdot \frac{(q^{\beta+1}-1)}{q-1} \cdot \frac{(r^{\gamma+1}-1)}{r-1} \ldots$.

(and of squares of divisors is $\frac{(p^{2\alpha+1}-1)}{p^2-1} \cdot \frac{(q^{2\beta+1}-1)}{q^2-1} \ldots\ldots$ etc.)

C9 (a) N is divisible by 3 (or 9) if sum of digits of N is divisible by 3 (or 9).
e.g. Let N be 'abcde' i.e. $10^4a + 10^3b + 10^2c + 10d + e =$
$9999a + 999b + 99c + 9d + (a + b + c + d + e)$
$= M(9) + (a + b + c + d + e)$
Then . H.S. $= M(3) \Leftrightarrow$ RHS $= M(3)$ i.e. $(a + b + c + d + e) = M(3)$ etc.
The rule of divisibility by 11, (a - b + c - d + e div. by 11), may be proved similarly:-
$N = 9999a + 1001b + 99c + 11d + (a - b + c - d + e)$. 7?

n	n^2	n^3	n^4
1	1	1	1
2	4	8	16
3	9	27	81
4	16	64	256
5	25	125	625
6	36	216	1296
7	49	343	2401
8	64	512	4096
9	81	729	6561
10	100	1000	10000
11	121	1331	14641
12	144	1728	20736
13	169	2197	28561
14	196	2744	38416
15	225	3375	50625
16	256	4096	65536
17	289	4913	83521
18	324	5832	104976
19	361	6859	130321
20	400	8000	160000

Squares. Last digits 1, 4, 9, 6, 5, 6, 9, 4, 1
same in 11-19 i.e. $x^2 \rightarrow (10+x)^2$ have same last digit
$x^2 \equiv (10+x)^2 \pmod{10}$.
Also symmetrical i.e. $x^2 \equiv (10-x)^2 \pmod{10}$.
'gaps' in sequence are 3, 5, 7, 9, ... i.e. 1st differences $(2n+1)$.

Cubes. Last digits repeat after 10 i.e. $x^3 \equiv (10+x)^3 \pmod{10}$.
gaps are 7, 19, 37, ... and 'gaps in gaps' - i.e. 2nd differences - are 12, 18, 24, 30....

Fourth powers. Last digits 1, 6, 1, 6, 5, 6, 1, 6, 1, 0 etc., symmetry and repetition after 10.
$x^4 \equiv (10+x)^4 \equiv (10-x)^4 \pmod{10}$, etc., etc.

C11.

p	y	Factor divisible by p: $2^{\frac{p-1}{2}}-1$	$2^{\frac{p-1}{2}}+1$
3	1		✓
5	2		✓
7	3	✓	
11	5		✓
13	6		✓
17	8	✓	
19	9		✓

Write $\frac{p-1}{2} = y$, so that $p = 2y+1$

$2^y \equiv 1$ for $p = 7, 17$ type $8k \pm 1$

$\equiv -1$ for $p = 3, 5, 11, 13, 19$ type $8k \pm 3$.

[It can be shown that $x^y \equiv 1$ or to -1 as x is or is not a quadratic residue to modulus p, and also that 2 is a quadratic residue of $8k \pm 1$ primes and a non-residue of $8k \pm 3$ primes. See Littlewood, D.E., A University Algebra, Heinemann 1950, p. 132.]

C11 (contd) Similarly 3 is a quadratic residue of primes of type $12k \pm 1$, {11, 13, 23, 37, 47...} and a non-residue of primes of type $12k \pm 5$, {5, 7, 17, 19, 29, 31, 41, 43....}. (See Littlewood, or Barnard S. and Child, J. M., Advanced Algebra, Macmillan, 1939, p.140).

		Factor divisible by p	
p	y	$3^y - 1$	$3^y + 1$
5	2		✓
7	3		✓
11	5	✓242	
13	6	✓728	
17	8		✓ 6562
19	9		✓ 19683
23	11	✓177146	
29	14		✓4782470

C12 (a) $15 = 7 + 8$; $2k+1 = k + (k+1)$.

(b) $14 \neq x + (x+1)$ but $14 = 2 + 3 + 4 + 5$

$10 = 1+2+3+4$

$18 = 3+4+5+6$

$22 = 4+5+6+7$

(c) e.g. $18 = 5+6+7$. $3k = (k-1)+k+(k+1)$ so every multiple of 3, except 3 itself, can be so expressed.

(d) Similarly every multiple of 5, from 15 upwards, can be expressed as the sum of 5 consecutive whole numbers (either $(k-2)+(k-1)+k+(k+1)+(k+2) = 5k$ or consider $15 = 1+2+3+4+5$

$20 = 2+3+4+5+6$ etc.)

But 2, 4, 8, 16 cannot be expressed as sums of consecutive integers (by trial).

(e,f) Theorem. Every number N can be expressed as a sum of 2 or more consecutive integers unless $N = 2^k$. No number of the form $N = 2^k$ can be so expressed. Proof:-

(A) N a multiple of $2k+1 \Rightarrow$ N expressible as sum of $2k+1$ consecutive integers, provided $N > k(2k+1)$ (as above for $N = 20, 2k+1 = 5$).

(B) $N = 2^p(2k+1) \Rightarrow$ N expressible as sum of 2^{p+1} consecutive integers provided $k \geq 2^p$; as above for $N = 14$:- k-1 k k+1 k+2 $\rightarrow 2(2k+1)$ (case k=3, p=1).

Hence(C) Unless $N = 2^p$, N is expressible as sum of consecutive integers. For let $N = 2^p(2k+1)$; if $k \geq 2^p$, use (B), otherwise use (A).

(D) If $N=2^p$ the expression is not possible.

For the sum of $(2\ell+1)$ consecutive integers is a multiple of $(2\ell+1)$ (and so $\neq 2^p$)
The sum of 2ℓ consecutive integers is ℓ times the sum of the middle two,

k k+1, and so, if this is 2^p, ℓ must be even.

But then $N/2 = 2^{p-1}$ can be similarly expressed (by retaining only the central 'block of ℓ numbers).
Continuing, we could prove that 2 cou'_ be expressed in this way - : contradiction.
(Alternatively, and quicker, since the sum of the middle two is $k+(k+1)$ which is odd, any such sum must contain an odd factor).

C13.

a	b	c	d	Product		
123	4567 x	76[illegible]	4321			4007
bd = 1973	4007				1973	
ad = 53	1483			53	1483	
bc = 349	3755			349	3755	
ac = 9	40[illegible]5			9	4095	
			Total	9	4497	7211	4007

Method

Work out (bd) Note 4007
$\div 10^4 \rightarrow$ 1973.4, store in M.
Work out (ad + bc) M+
RCL (402) 7211.4
Note 7211
$\div 10^4 \rightarrow$ 402.7, store M
Work out (ac) M+
RCL → 94497.7. Note 94497

By using 10^4 as 'base', calculations involving 4n digits, for any n, can, in principle, be performed - useful for evaluating large powers and for number - theoretic calculations. (Similar methods can be used on a computer - e.g. for calculation of π to large number of decimal places.)

C14. We have $\frac{1}{31} = 0.\overbrace{03225\ 80645\ 16129}^{\lambda = 15}\ldots$ $x_{n+1} = x_n(2-31x_n)$

$x_0 = 0.03$

$31.x_0$	93	9951	99997599
( )	107	10049	100002401
x_1	321	3225729	32258064 80

Multipliers are $1+\varepsilon$ with $\varepsilon =$ → 7×10^{-2}, 5×10^{-3} (or 49×10^{-4}), 2.4×10^{-5}

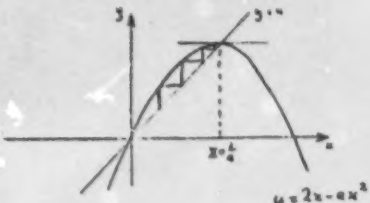
C14(cont.) Conjecture $\varepsilon_{n+1} = \varepsilon_n^2$.

$\varepsilon_n = 1 - ax_n$, so $x_n = \frac{1-\varepsilon_n}{a}$ and x_n too small $\Rightarrow \varepsilon_n$ too big

Proof. Now $x_{n+1} = x_n(1+\varepsilon_n)$

$$\frac{1-\varepsilon_{n+1}}{a} = \frac{1-\varepsilon_n}{a} \cdot (1+\varepsilon_n), \Rightarrow \varepsilon_n^2 = \varepsilon_{n+1}.$$

Graphically:-



In the neighbourhood of the solution, the curve is like $y=x^2$ at the origin - and distance from $x=\frac{1}{a}$ is squared at each stage.

C14. For $\frac{1}{41}$, $x_0 = 0.025$ | $x_1 = 0.025 \times 0.975 = 0.24375$ | $x_2 = 24390234375$

$41.x_0 = 1.025$	$41x_1 = 999375$	$41x_2 = 999999609375$
$(\ ) = 0.975$	$(\ ) = 1.000625$	$(\ ) = 1.000000380625$

$x_3 = 24390243\overline{65}\ldots$

We have:-

$\varepsilon_1 = -2.5 \times 10^{-2}$

$\varepsilon_2 = 6.25 \times 10^{-4}$

$\vdots$

similarly

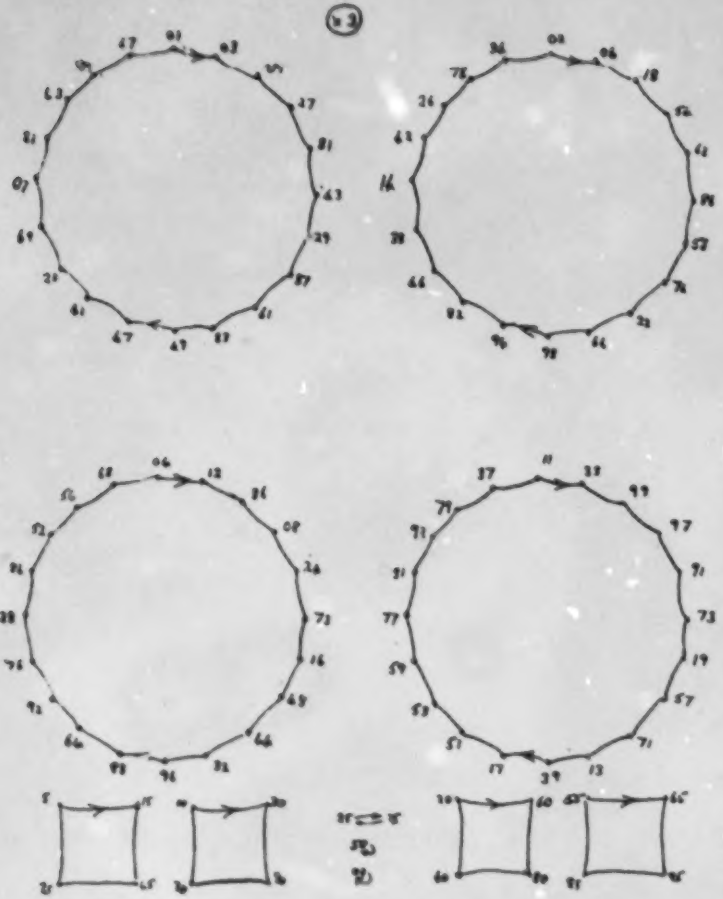
$(\frac{1}{41} = 0.02439\overline{02439}.....)$

For $\frac{1}{17}$, take $x_0 = 0.05$ | $x_1 = 0.05 \times 1.15 = 0.0575$ | $x_2 = 0.0575 \times 1.0225 = 0.05879375$ | $x_3 = 058823\overline{51}$

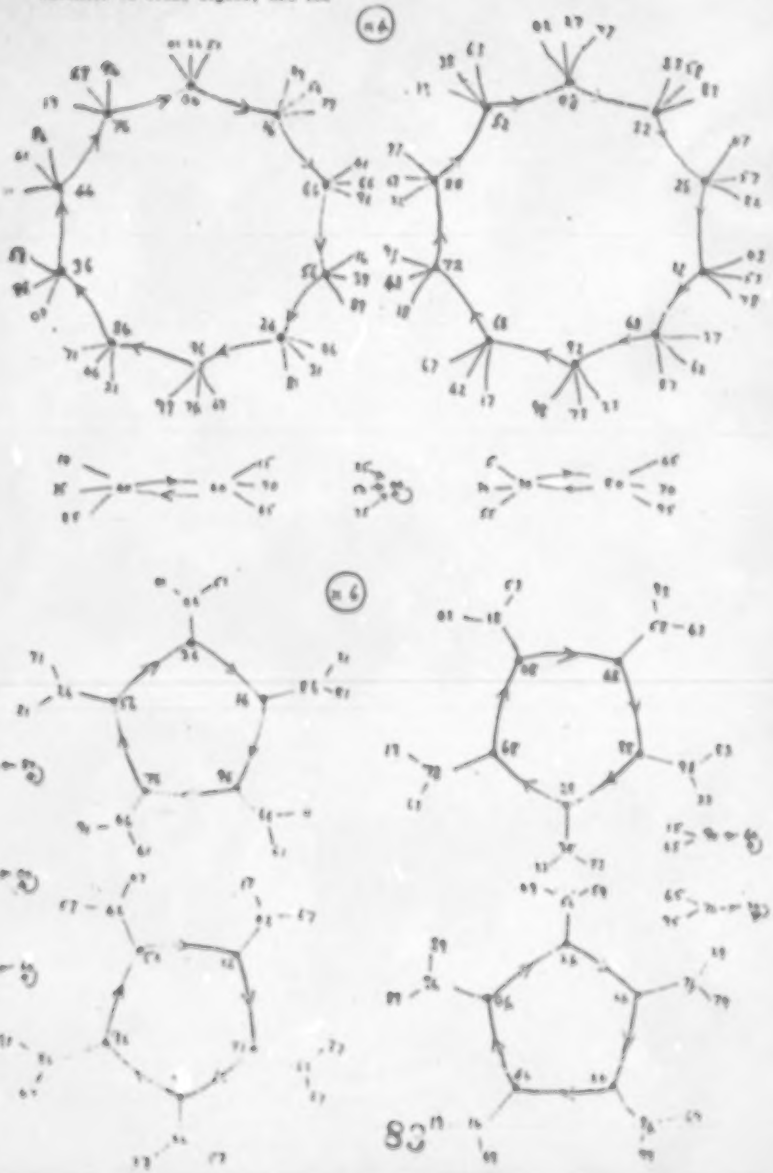
$17x_0 = 0.85$	$17x_1 = 0.9775$	
$(\ ) = 1.15$	$(\ ) = 1.0225$	

$(\frac{1}{17} = 0.0588235294117647\overline{0588}\ldots.)$

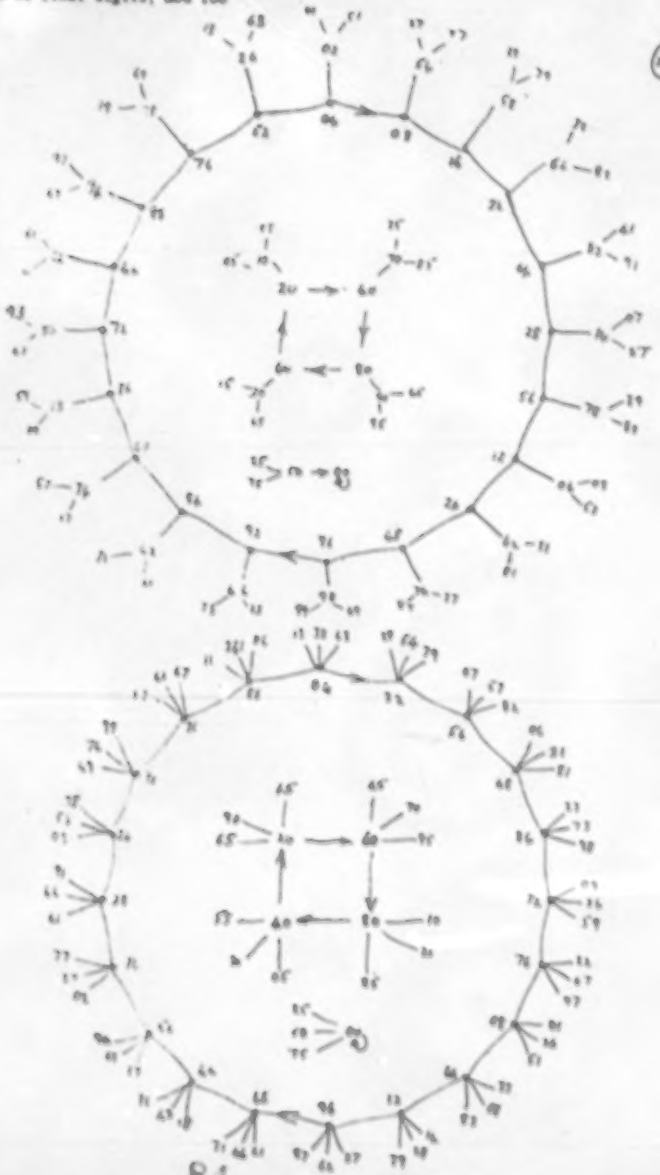
Patterns of final digits, mod. 100 (see C7)



Patterns of final digits, mod 100

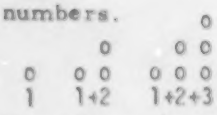


Patterns of final digits, mod 100



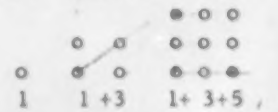
Patterns of final digits, mod 100

C1. (a) The numbers 1, 3, 6, 10, 15, 21, ... are triangular numbers. Find a formula for the nth triangular number.



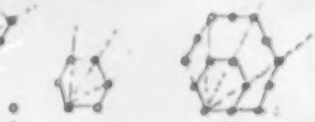
The first few numbers are odd, odd, even, even, odd, odd, ...
How does the pattern continue? Explain.

(b) Other series are the square numbers 1, 4, 9, 16, ...



the pentagonal numbers 1, 5, 12, ..

the hexagonal numbers 1, 6, 15, ...



(i) From the square numbers it appears that $1 + 3 + 5 + 7 + 9$ is a perfect square. etc. Check this. Explain.

(ii) Add up any two consecutive triangular numbers. Can you explain the result? (Hint: use a Δ diagram for each of the triangular numbers)

(iii) Find all the pentagonal numbers less than 100. Find a formula for the nth pentagonal number. Check it.

(iv) Find a formula for the nth hexagonal number. Explain the connection with triangular numbers.

(v) Find a formula for the nth 'X'-agonal number.

(vi) (*) 36 is a Δ no. and also a square number. Are there any others?

C2. (a) (i) The product of any 2 consecutive integers is even.

(ii) The product of three consecutive integers is a multiple of 6.

(iii) The product of four consecutive integers is a multiple of 24,

and (iv) is also one less than a perfect square.

(b) (i) $1^3 + 2^3 = 9$

$1^3 + 2^3 + 3^3 = 36$

Continue this pattern. State a rule and test it in some more cases. Try to prove the rule.

(ii) (Harder) Can you find a rule for calculating $1^2 + 2^2 + 3^2 + \ldots + n^2$?

(The first few values are

n	1	2	3	4	5	6
Sum	1	5	14	30	55	91...

)

(Hint: You may find it easier to consider 6 x sum).

Copyright © F.R. Watson, University of Keele, 1979

Note. The reproduction of Pupil material by teachers for use with their own pupils is expressly permitted.

C3. The number 13 can be expressed as the sum of two squares $13 = 3^2 + 2^2$
So can 17 ($= 4^2 + 1^2$) and 25 ($= 5^2 + 0^2 = 4^2 + 3^2$) and 65 $= (8^2 + 1^2 = 7^2 + 4^2)$
But there are no whole numbers x, y such that $19 = x^2 + y^2$ (check).

(a) Choose some more numbers and test whether they can be expressed in the form $N = x^2 + y^2$.

(b) Classify the values of N according to their remainders when divided by 4. How is this related to (a)?
(Hint: x, y are of the form 2k or 2k+1).

(c) Let primes of type 4k+1 be of type 'p' (e.g. 17 is of type p)
and primes of type 4k+3 be of type 'q' (19 is of type q)

Can you make a statement about the cases:-

(i) N is of type p

(ii) N is of type q (Prove it!)

(iii) N is composite?

C4. (3, 4, 5) is a Pythagorean triple (since $3^2 + 4^2 = 5^2$, so that a triangle with sides 3, 4, 5 is right-angled by the converse of Pythagoras' theorem).
(6, 8, 10) is also such a triple, but it is obtained by doubling all the members of the triple (3, 4, 5) so it isn't really a 'new' one.

(a) Find some more Pythagorean triples which are primitive (that is they have no common factor; these are the only ones we shall consider).

(b) Consider those triples in which the largest side is one more than the next side. Arrange them in a sequence.
The general form is (x, n, n+1): the first few are

3 4 5
5 12 13
7 24 25
. . . . . .

(c) Can you find any triple not in this series? In particular, can you find other triples of the type (x, n, n+2)? - (4, 3, 5) is one.

(d) Is it possible for two numbers in a primitive triple to have a common factor? (Proof?)
How many numbers in a primitive triple are odd? Why?
Is the largest number in a primitive triple odd or even? Why?

(e) "The product of the numbers in a primitive triple is always a multiple of N" Examine some primitive triples and suggest a value for N.

(f) (Hard) It can be proved that every primitive triple is of the form $(\ell^2 - m^2, 2\ell m, \ell^2 + m^2)$ where ℓ, m have no common factor, and just one is even. Assuming this result, prove your conclusion in (e).

C5. Investigate the Fibonacci sequence $(1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \ldots\ldots)$
$(u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \quad u_7 \ldots\ldots)$

For example,

(a) $1 + 1 + 2 + 3 + 5 = 12$ —— find other similar statements and make a conjecture.

(b) $1 + 2 + 5 + 13 = 21$

(c) $1^2 + 1^2 + 2^2 + 3^2 + 5^2 = 40$

(d) "$u_{n+k} = u_{n-1}\, u_k + u_n\, u_{k+1}$". Is this true? (If so prove it)

(e) $1 + 3 + 8 + 21 = 33$

(f) $8^2 - 3^2 = 55$

(g) $\frac{1}{2}(13 - 3) = 5$

(h) Note that $u_3 = 2$. What do you observe about u_6, u_9, u_{12}.....?
Consider similarly $u_4(=3)$, and u_8, u_{12}, u_{16}.....

and $u_5(=5)$, and u_{10}, u_{15}, u_{20}...

State a general conjecture concerning u_s and u_{rk}.

(contd....)

C6. (a) Find $\frac{1}{7}$ as a recurring decimal. Find $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, $\frac{6}{7}$

(b) Find $\frac{1}{13}$, $\frac{2}{13}$... up to $\frac{12}{13}$. Arrange these into two sets, according to the digits which appear in the repeating portion.

(c) How many sets of decimal expansions arise in this way from $\frac{n}{39}$ $(1 \leq n \leq 38)$?, and from $\frac{n}{41}$ $(1 \leq n \leq 40)$?

(d) Why do some fractions give recurring decimals, whilst others give terminating or 'mixed' recurring decimals? (Try $\frac{1}{k}$ for the following values of k:- 2, 3, 4, . . . , 16, 20, 25.)

(e) The 'period length', λ, of a recurring decimal is the number of figures in the repeating sequence of digits in its decimal expansion. (e.g. the period length for $\frac{1}{7}$ is 6, and for $\frac{1}{41}$ is 5).
If λ is the period length for $\frac{1}{p}$, where p is a prime, what is the connection between λ and p?

(contd....)

Copyright © F.R. Watson, University of Keele, 1979

C5. contd.

(i) Find the hcf of 2 adjacent Fibonacci numbers.
Find the hcf of u_n and u_{n+2} for several values of n.

(j) hcf (u_m, u_n) is always a Fibonacci number. Which one?

(k) Use (j), (assumed true, if need be), to prove

"If u_m is a factor uf u_n then m is a factor of n."

(l) If each number in the sequence is divided by 2, the remainders are 1 1 0 1 1 0...
Division by 3 gives 1 1 2 0 2 2 1 0 1 1 2 0 2 2 1 0...
Investigate division by 7, and by other numbers. Can the sequence of remainders go on for ever without repeating?
If the sequence for division by n is periodic, how does the length of the period depend on n?
(To prove the last result is very difficult - but you should try to prove or explain the other properties which you observe in (a) to (l)).

C6. contd.

(f) The numbers 1, 5, 7, 11 are the only numbers less than 12 which have no factor (>1) in common with 12. There are 4 of them - we say $\phi(12) = 4$, where $\phi(n)$ is the number of integers j, $1 \leq j \leq n-1$, which have no factor in common with n.
($\phi(n)$ is Euler's function).

Find $\phi(n)$ for $2 \leq n \leq 20$
What is $\phi(p)$ when n=p, a prime?
What can you say about $\phi(p^2)$?; about $\phi(pq)$ where p, q are distinct primes?
What can you say about $\phi(ab)$?

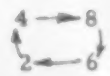
(g) What can be said about the period length of a fraction $\frac{1}{n}$, where n is not prime?

C7. **Final digits.**

(a) Take any number ending in 9 and multiply it by 7. What is its last digit? We can write this as $9 \xrightarrow{\times 7} 3$

("when we multiply by 7, a final digit of 9 changes to a final digit of 3").

(b) What happens to a number ending in 3? 4? What do you think this chain means?



(c) Examine all the possible final digits and make other chains (for multiplication by 7).

(d) What patterns in final digits are there for multiplication by 3? (Do you notice anything? Can you guess why?)

(e) Now try other multiplying numbers (0, 1, 5 are not very interesting).

(f) Look for similar chain patterns in the last <u>two</u> digits (using 7 as a single digit multiplier).

(g) Try other single digit multipliers, looking for chains in the last two digits of the products.

(h) Try to explain the chains in (b) and (c) above. (Hint: 7x7x7x7=2401) Likewise, try to explain the other patterns (for multiplication by 3, 2, 4, 6....).

(i) <u>Harder</u>. Use the fact that $7^4 = 2401$ and $3^4 = 81$ to explain the behaviour of chains of pairs of final digits in (f) and (g) - in particular why 7 and 3 behave differently for <u>pairs</u> of final digits, but similarly for <u>single</u> final digits.

C8. (a) The number 12 is divisible by 1, 2, 3, 4, 6, 12 - so that it has 7 divisors. On the other hand, 13 is divisible by 1 and 13 only, so that it has two divisors. Make a table showing the divisors of each of the numbers from 1 to 50.

(b) Make a table showing the <u>number</u> of divisors of each of the numbers 1 - 50. Can you see any regularities in the table? Can you devise a <u>rule</u> for the number of divisors of a number N? (This is rather hard. Start with easy cases first e.g. 31, .. 25, .. 35) The final objective is to state a formula for the number of divisors of $N = p^a q^b r^c \ldots$ where p, q, r are the prime factors of N)

(c) Investigate the <u>sums</u> of divisors of N similarly, and try to find a formula. (<u>Hint</u>: again, consider the easy cases first. You may find it useful to remember that

$$\frac{x^n - 1}{x - 1} = 1 + x + x^2 + \ldots\ldots + x^{n-1}).$$

Copyright © E.R. Watson, University of Keele, 1979

C9. (a) Which of the following numbers are divisible by 3?

$\{879,\ 3762,\ 28795,\ 7826,\ 2311,\ 6134,\ 9859,\ 7290\}$

(b) Which of the numbers in (a) are divisible by 9?

(c) If you do not know the rule for divisibility by 3 and by 9, make a table as follows:-

Number	Sum of digits	Divisible by 3?	by 9?
879	8 + 7 + 9 = 24	yes	no
3762	18	yes	yes
....	.....	...	....

Can you see any connection between divisibility by 3 or 9 and the sum of the digits?

(d) Try to prove the rule referred to in (c)
(Hint: Start with the easy case of a 3 (or 2) digit number. "abc" = 100a + 10b + c. The sum of digits is (a + b + c)
Notice that 10 = 9 + 1, and 100 = 99 + 1).

C10. Make a table of squares (cubes, fourth powers) of the numbers from 1 - 20. What regularities occur in the table and how may they be explained?

C11. If p is a prime number, $2^{p-1} - 1$ is always a multiple of p. (Check this)
(Fermat's theorem: see Series B, no. 6).
Thus either $(2^{p-1} - 1)$ or $(2^{p-1} + 1)$ is divisible by p. Find a rule for deciding which.
Now repeat the investigation for $3^{p-1} - 1$.

C12. (a) 15 can be written as the sum of the two consecutive whole numbers $15 = 7 + 8$. Show how this can be done with any odd number.

(b) Can you write 14 as the sum of two consecutive whole numbers? Of 4 consecutive whole numbers?
Try other even numbers 10, 18, 22,.....

(c) "Any multiple of 3 can be written as the sum of 3 consecutive whole numbers". Is this true? (Test it. Prove your statement if you can).

(d) $15 = 1 + 2 + 3 + 4 + 5 = 4 + 5 + 6 = 7 + 8$. Thus 15 can be written in several ways as the sum of consecutive whole numbers.
6 can be written in only one way $6 = 1 + 2 + 3$.
Can every number be written in some way as the sum of consecutive whole numbers?

(e) Show that the sum of n consecutive whole numbers is a multiple of n if n is odd, and a multiple of n/2 if n is even.
What does this imply about the representation of 2^p as a sum of consecutive integers, where p is a positive whole number?

(f) Show that every whole number not of the form 2^p can be expressed as a sum of consecutive whole numbers.

C.13 Use the identity $(ax + b)(cx + d) = acx^2 + (bc + ad)x + bd$, where $x = 10^4$, to perform 1234567 x 7654321 on a machine with capacity 8 digits.
Can you find a way of performing the "carry" 'automatically'?

C.14 The sequence $x_{n+1} = x_n(2 - ax_n)$ generates approximations to the reciprocal $\frac{1}{a}$. Use it to find $\frac{1}{31}$ by iteration ($x_0 = 0.03$). Compare with the (recurring decimal) exact answer.
At each stage of the calculation let $(2 - 31x_n) = 1 + \varepsilon_n$, so that $\varepsilon_n = 31x$ (error in x_n). What do you notice about ε_n and about the relationship of successive values of ε_n?
Test your conjectures in finding $\frac{1}{41}$ ($x_0 = 0.025$), $\frac{1}{17}$, $\frac{1}{19}$, $\frac{1}{39}$, etc.
Can you prove them?

Copyright © F.R. Watson, University of Keele, 1979

Exploring Numbers - Number Patterns and Investigations

Pupil's Activities. (Series C)

(Some suggestions for using the electronic calculator).

(A set of Teachers' notes and solutions is available)

Copyright © F.R. Watson, University of Keele, 1979

Note. The reproduction of Pupil material by teachers for use with their own pupils is expressly permitted.